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Nonexistence of periodic solutions in delayed Lotka–Volterra systems

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1. Introduction

In this paper we derive sufficient conditions for the nonexistence of nonconstant periodic solutions of Volterra differential equations with distributed delays where the delay kernels are chosen among γ-functions or their suitable convex normalized combinations. The reason of this choice for the kernels is that the Volterra delay differential equations can thus be transformed in an expanded system of ordinary differential equations by the standard "linear chain trick" method [7]. To this expanded o.d.e. Volterra system we can apply the conditions, encoded by the logarithmic norm of some Jacobian related matrix, that Li and Muldowney [5] have obtained for the nonexistence of (nontrivial) periodic solutions for autonomous ordinary differential equations in $\mathbf{R}^N$, conditions that generalize to the case $N > 2$ the Bendixon and Dulac critera.

The general structure of the o.d.e. systems obtained from Volterra differential delay systems (when the delay kernels are convex normalized combinations of γ-functions) has been studied, mainly in relation to boundedness and existence of an equilibrium and

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its global asymptotic stability, in some papers like Solimano and Beretta [9], Beretta and Solimano [2] in which the authors considered the "linear" generalization of the Volterra delay differential equations and in a paper by Beretta et al. [1] where the linear generalization of Volterra delay equations was applied both to a prey–predator model with prey's shelter and to an SIR epidemic model with incubation time. In all these last three papers boundedness and global stability (and hence nonexistence of periodic solutions) was encoded on a suitable extended "community matrix" by requiring that it belongs to the S_w class. We show that the sufficient conditions for non-existence of periodic solution obtained by applying the Li and Muldowney's criteria include (i.e. are more general) the case of S_w community matrices.

The paper is organized as follows: in Section 2 we recall the main criteia by Li and Muldowney [5] and some related results on stability matrices. Moreover, we recall the general structure of Volterra expanded o.d.e. systems obtained from Volterra differential delay systems and we report some results about boundedness of their solutions and existence of a globally asymptotically stable nonnegative equilibrium. In Section 3 we first derive the boundedness properties and then suitable permanence or persistence results of solutions in relation to applicability of Li and Muldowney criteria. Hence in Section 4 we consider the application of Li and Muldowney criteria deriving the sufficient conditions for the non-existence of nontrivial periodic solutions. Finally, Section 5 with discussion of the results concludes the paper.

2. General results

The Volterra delay differential systems with distributed delays can be written as

$$\dot{x}_i = x_i\left(e_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} \gamma_{ij}\int_{-\infty}^{t} f_{ij}(t-u)x_j(u)\,\mathrm{d}u\right),$$

$$i \in \mathbf{N} \triangleq \{1,2,\ldots,n\}, \tag{2.1}$$

where for each $\gamma_{ij} \neq 0$, $f_{ij} : [0,+\infty) \to \mathbf{R}$ are continuous nonnegative functions obtained by convex combination

$$f_{ij}(u) = \sum_{k=1}^{p_{ij}} c_{ij}^{(k)} f_{ij}^{(k)}(u), \quad c_{ij}^{(k)} \geqslant 0, \quad \sum_{k=1}^{p_{ij}} c_{ij}^{(k)} = 1 \tag{2.2}$$

of functions which are solutions of linear differential equations with constant coefficients:

$$f_{ij}^{(k)}(u) = \frac{\alpha_{ij}^{k}}{(k-1)!}u^{k-1}\exp(-\alpha_{ij}u), \quad \alpha_{ij} \in \mathbf{R}_{+}, \quad k \in \{1,2,\ldots,p_{ij}\} \tag{2.3}$$

and satisfy the normalized condition

$$\int_{0}^{+\infty} f_{ij}(u)\,\mathrm{d}u = 1.$$

We remind that the average time delay of (2.3) is $T = k/\alpha_{ij}$. We refer to (2.3) as to a γ-distribution (or γ-function) of order k. According to linear chain trick (see [7] or [11]) we put

$$x_{ij}^{(k)}(t) := \int_{-\infty}^{t} f_{ij}^{(k)}(t-u)x_j(u)\,\mathrm{d}u, \quad k = 1,\ldots,p_{ij},$$

$$x_{ij}^{(0)}(t) := x_j(t), \quad i,j \in \mathbf{N}, \quad \gamma_{ij} \neq 0. \tag{2.4}$$

Let "p" the number of distinct functions $x_{ij}^{(k)}$ and $P = \{n+1,\ldots,n+p\}$ the set of all their indices. According to (2.4), system (2.1) is transformed in an expanded system of "$n+p$" ordinary differential equations

$$\dot{x}_i = x_i\left(e_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n}\gamma_{ij}\sum_{k=1}^{p_{ij}} c_{ij}^{(k)}x_{ij}^{(k)}\right), \quad i \in \mathbf{N},$$

$$\dot{x}_{ij}^{(k)} = \alpha_{ij}x_{ij}^{(k-1)} - \alpha_{ij}x_{ij}^{(k)}, \quad k = 1,\ldots,p_{ij}, \quad i,j \in \mathbf{N} : \gamma_{ij} \neq 0, \tag{2.5}$$

where the last "p" are linear differential equations with real constant coefficients. By introducing the vector

$$X = \begin{pmatrix} X^{(n)} \\ X^{(p)} \end{pmatrix} \in \mathbf{R}_{+0}^{n+p},$$

where $X^{(n)} = (x_1,\ldots,x_n)^{\mathrm{T}}$ and $X^{(p)} \in \mathbf{R}_{+0}^{p}$ is the vector of p-functions (2.4), system (2.5) can be rewritten in the following matrix form:

$$\dot{X} = \begin{pmatrix} \mathrm{diag}(X^{(n)}) & 0 \\ 0 & I_p \end{pmatrix}(e + AX), \tag{2.6}$$

where $e = (e_1,\ldots,e_n,0,\ldots,0)^{\mathrm{T}} \in \mathbf{R}^{n+p}$, I_p is the "$p \times p$" identity matrix and $A = (\tilde{a}_{ij})_{i,j\in\mathbf{N}\cup P}$ is the $(n+p)\times(n+p)$ real constant matrix of coefficients of (2.5) (if $i,j \in \mathbf{N}$, $\tilde{a}_{ij} = a_{ij}$, $\tilde{a}_{ij} = \gamma_{ij}c_{ij}^{(k)}$ if $i \in \mathbf{N}, j \in P$ etc.) The initial conditions for (2.1) require the knowledge in the past of the nonnegative, continuous and bounded functions

$$x_i(u) = \varphi_i(u), \quad u \in (-\infty, 0] \quad \text{for all } i \in \mathbf{N}. \tag{2.7}$$

The (2.7) provide the i.c. for (2.5) or (2.6). In fact

$$\dot{x}_i(0) = \varphi_i(0), \quad i \in \mathbf{N},$$

$$x_{ij}^{(k)}(0) = \int_{-\infty}^{0} f_{ij}^{(k)}(-u)\varphi_j(u)\,\mathrm{d}u, \quad k = 1,\ldots,p_{ij}, \quad i,j \in \mathbf{N}, \tag{2.8}$$

i.e. $X(0) \in \mathbf{R}_{+0}^{n+p}$.

Denote by

$$x(t) = x(t,\varphi) = \mathrm{col}(x_1(t,\varphi),\ldots,x_n(t,\varphi)), \tag{2.9}$$

a solution of (2.1) with i.c. (2.7) for $t \geqslant 0$ and consider the p-functions (2.4)

$$x_{ij}^{(k)}(t) = \int_{-\infty}^{t} f_{ij}^{(k)}(t-u)x_j(u)\,\mathrm{d}u, \quad k = 1,\ldots,p_{ij}; \;\; i,j \in \mathbf{N},$$

where $x_{ij}^{(0)}(t) = x_j(t)$. Then the vector function

$$z(t) = \mathrm{col}(x_1(t),\ldots,x_n(t); x_{ij}^{(k)}(t), k = 1,\ldots,p_{ij}; \;\; i,j \in \mathbf{N}), \tag{2.10}$$

where the first n-components are solution of (2.1), is a solution of the expanded system of o.d.e. (2.5) with i.c. (2.8). Now, we can prove the following Lemma 2.1.

Lemma 2.1. *Assume that* (2.1) *with i.c.* (2.7) *and delay kernels* (2.2), (2.3) *has a periodic solution* $x(t) = x(t,\varphi)$ *with some period* $T > 0$, *i.e.*, $x(t+T) = x(t)$ *for any* $t \geqslant 0$. *Then the vector function* $z(t)$ *in* (2.10) *is a T periodic solution of the expanded system of o.d.e.* (2.5) *with i.c.* (2.8).

Proof. It is enough to consider the "p" functions $x_{ij}^{(k)}(t)$ in (2.10):

$$x_{ij}^{(k)}(t) = \int_{-\infty}^{t} f_{ij}^{(k)}(t-u)x_j(u)\,\mathrm{d}u$$

and the change of integration variables: $t - u = s$. We get

$$x_{ij}^{(k)}(t) = \int_{0}^{+\infty} f_{ij}^{(k)}(s)x_j(t-s)\,\mathrm{d}s.$$

Of course, if the "n" variables $x_j(t)$ are T periodic, then

$$x_{ij}^{(k)}(t+T) = \int_{0}^{+\infty} f_{ij}^{(k)}(s)x_j(t+T-s)\,\mathrm{d}s = \int_{0}^{+\infty} f_{ij}^{(k)}(s)x_j(t-s)\,\mathrm{d}s = x_{ij}^{(k)}(t),$$

thus implying that the vector function (2.10) is also T periodic.

This trivial result implies:

Corollary 2.2. *Assume that the expanded system of o.d.e.* (2.5) *cannot have non-constant periodic solutions. Then also the integro-differential system* (2.1) *with delay kernels* (2.2), (2.3) *cannot have nonconstant periodic solutions.*

Proof. If (2.1) has some T periodic solution, then also (2.5) must have a T periodic solution in contradiction with the assumption.

The result in Lemma 2.1 or Corollary 2.2 makes meaningful to apply Li and Muldowney criteria to the expanded system of o.d.e. (2.5) (or (2.6)) to infer the non-existence of non-constant periodic solutions for (2.1) in the assumption that the delay kernels in (2.1) are chosen as in (2.2), (2.3).

For the expanded autonomous o.d.e. system (2.6) the following results were proven (see [2,9]):

Theorem 2.3. *If* $-A \in S_w$, *then all the solutions of* (2.6) *with initial condition* $X(0) \in \mathbf{R}_{+0}^{n+p}$ *are bounded. Furthermore a compact subset* $\Omega \subset \mathbf{R}_{+0}^{n+p}$ *exists containing the* ω*-limit set of* (2.6). *The* ω*-limit set is nonempty because of the existence of at least* *one nonnegative equilibrium* X^* *of* (2.6). *If* X^* *is positive then* $X^* \in \mathring{\Omega}$ (*the interior of* Ω).

Theorem 2.4. *If* $-A \in S_w$, *then system* (2.6) *has a nonnegative equilibrium, say* X^*, *which is globally asymptotically stable with respect to*

$$\mathbf{R}_I^{n+p} = \{X \in \mathbf{R}_{+0}^{n+p} | x_i > 0 \quad if\ i \notin I\},$$

where $I \subset N \cup P$ *is the set of indices such that* $x_i^* = 0$.

We recall the definition of an S_w matrix [10]:

Definition 2.5. Let $Q = (q_{ij})$ be an $n \times n$ real matrix. $Q \in S_w$ means that a diagonal real positive matrix W exists such that $WQ + Q^{\mathrm{T}}W$ is positive definite.

Thus, to the autonomous o.d.e. system (2.6) the following criteria by Li and Muldowney [5] can be applied. Consider the general system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x), \tag{2.11}$$

where $F(x) \in \mathbf{R}^N$, $x \mapsto F(x)$ is C^1 in an open subset D_0 of $\mathbf{R}^N$. Denote by $J = (\partial F/\partial x)$ the Jacobian of (2.11) and by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ the eigenvalues of $(\frac{1}{2})[(\partial F/\partial x) + (\partial F/\partial x)^{\mathrm{T}}]$. Denote by $J^{[2]}$ the $\binom{N}{2} \times \binom{N}{2}$ matrix which is the second additive compound matrix associated to the Jacobian matrix J (see [5,6], also the appendix for definition) and remind that if $x \in \mathbf{R}^N$ then the corresponding logarithmic norms of $J^{[2]}$ (that we denote by $\mu(J^{[2]})$) endowed by the vector norms (i) $|x|_1 = \sum_i |x_i|$, (ii) $|x|_\infty = \sup_i |x_i|$ and (iii) $|x|_2 = (x^{\mathrm{T}}x)^{1/2}$, respectively are:

(i) $\mu_1(J^{[2]}) = \sup\left\{\frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{j \neq r,s}\left(\left|\frac{\partial F_j}{\partial x_r}\right| + \left|\frac{\partial F_j}{\partial x_s}\right|\right) : 1 \leqslant r < s \leqslant N\right\}$,

(ii) $\mu_\infty(J^{[2]}) = \sup\left\{\frac{\partial F_r}{\partial x_r} + \frac{\partial F_s}{\partial x_s} + \sum_{j \neq r,s}\left(\left|\frac{\partial F_r}{\partial x_j}\right| + \left|\frac{\partial F_s}{\partial x_j}\right|\right) : 1 \leqslant r < s \leqslant N\right\}$,

(iii) $\mu_2(J^{[2]}) = \lambda_1 + \lambda_2$;

where $\mu_\infty(J^{[2]}) < 0$ implies the diagonal dominance by row of the matrix $J^{[2]}$ and $\mu_1(J^{[2]}) < 0$ means its diagonal dominance by column. Then the following holds:

Theorem 2.6. *A simple closed rectifiable curve which is invariant with respect to* (2.11) *cannot exist if* $\mu(J^{[2]}) < 0$ *or* $\mu(-J^{[2]}) < 0$ *on* $\mathbf{R}^N$, *where* μ *is one of the logarithmic norms.*

Let D_0 be a simple connected subset of $\mathbf{R}^N$ and $A(x)$, $x \in \mathbf{R}^N$, be a C^1 nonsingular $\binom{N}{2} \times \binom{N}{2}$ real valued matrix function on D_0. Furthermore, denote by A_F the matrix obtained from A by replacing each entry $a_{ij}(x)$ of A by

$$(a_{ij})_F = \left(\frac{\partial a_{ij}(x)}{\partial x}\right)^{\mathrm{T}} \cdot F(x) = \sum_{k=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_k} F_k(x), \tag{2.12}$$

i.e. A_F is the directional derivative of A in the direction of F. Assume that the solutions of (2.11) exist for all $t \geqslant 0$. A subset D_1 of D_0 is said to be "absorbing" with respect to (2.11) if each bounded subset D of D_0 satisfies $x(t,D) \subset D_1$ for all sufficiently large t. The following result holds [5]:

Theorem 2.7. *Assume that*
(a) D_0 *is simply connected*;
(b) $\mu(A_F A^{-1} + AJ^{[2]}A^{-1}) \leqslant b < 0$ *on a set* D_1 *which is absorbing with respect to* (2.11).
Then there is no simple closed rectifiable curve in D_0 *which is invariant with respect to* (2.11).

Assume that $\Omega \subset \mathbf{R}^N$ is a compact global attractor for (2.11). Then the following holds also true:

Corollary 2.8. *If* $\Omega \subset \mathbf{R}^N$ *is a compact global attractor of* (2.11) *on which*

$$\mu(A_F A^{-1} + AJ^{[2]}A^{-1}) < 0 \tag{2.13}$$

for some logarithmic norm then in Ω *there is no simple closed rectifiable curve which is invariant with respect to* (2.11).

Remark that if we choose A as a real constant and nonsingular matrix, then $A_F = 0$ and condition (2.13) becomes $\mu(AJ^{[2]}A^{-1}) < 0$ on Ω for some logarithmic norm. Finally, if $A = I$, the identity $\binom{N}{2} \times \binom{N}{2}$ matrix then (2.13) reads $\mu(J^{[2]}) < 0$ on Ω.

It may be interesting to recall also the following result by Li and Wang [6]. Assume A is an N-dimensional real matrix and denote by $\sigma(A) = \{\lambda_i : i = 1, 2, \ldots, N\}$ its spectrum and by $s(A)$ its "stability modules":

$$s(A) = \max\{\mathbf{Re}\,\lambda \colon \lambda \in \sigma(A)\}. \tag{2.14}$$

We say that matrix A is stable if $s(A) < 0$. Then, the following holds [6]:

Theorem 2.9. *Assume that* $(-1)^N \det(A) > 0$. *Then* A *is stable if and only if* $\mu(A^{[2]}) < 0$ *for some logarithmic norm* μ.

Hence, Theorem 2.9 is the link between negative criteria for existence of periodic solutions and stability of Jacobian matrix. Furthermore, assume that $A(\alpha)$ is an $N \times N$ real matrix depending with continuity on some real parameter $\alpha \in (a,b) \subset \mathbf{R}$. Then $\alpha_0 \in (a,b)$ is said to be a "Hopf bifurcation point" for $A(\alpha)$ if $A(\alpha)$ is stable for $\alpha < \alpha_0$, and there exists a pair of complex eigenvalues $\mathbf{Re}\,\lambda(\alpha) \pm \mathbf{Im}\,\lambda(\alpha)$ of $A(\alpha)$ such that $\mathbf{Re}\,\lambda(\alpha) > 0$ while the other eigenvalues of $A(\alpha)$ have non-zero real parts for $\alpha > \alpha_0$. Then we get the following (see [6]):

Corollary 2.10. *No Hopf bifurcation points of* $A(\alpha)$ *exist in* (a,b) *if* $\mu(A^{[2]}(\alpha)) \leqslant 0$ *for some logarithmic norm* μ *and all* $\alpha \in (a,b)$.

Of course $A^{[2]}$ in Theorem 2.9 and Corollary 2.10 is the $\binom{N}{2} \times \binom{N}{2}$ second additive compound matrix associated with the matrix A.

3. Two-dimensional Volterra systems with 2 delays

Now let us consider an n-dimensional Volterra delay differential systems with distributed delays expressed by (2.1) with delay kernels (2.2) and (2.3). The systems can be expressed as (2.5) by using p new variables (2.4) and become $(n+p)$-dimensional o.d.e. Their Jacobian has a size $(n+p)\times(n+p)$ and its second additive compound, is $\binom{n+p}{2} \times \binom{n+p}{2}$. Hence, in the following we restrict our systems with $n=2$ and $p \leqslant 2$, that is, we consider two-dimensional Volterra systems with at most 2 delays, whose kernels are given by the first or second order γ-distributions ($k = 1$ or 2 in (2.3)). Hereafter, for the simplicity of notation, we denote $x_{ij}^{(k)}$ as $x_j^{(k)}$.

Because of the symmetry of the systems, they are described as follows:

- a system with one first order delay:

$$\begin{aligned}
\dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma x_j^{(1)}),\\
\dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2),\\
\dot{x}_j^{(1)} &= \alpha x_j - \alpha x_j^{(1)}, \quad j = 1 \text{ or } 2.
\end{aligned} \tag{3.1}$$

- a system with one-second order delay:

$$\begin{aligned}
\dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma x_j^{(2)}),\\
\dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2),\\
\dot{x}_j^{(1)} &= \alpha x_j - \alpha x_j^{(1)},\\
\dot{x}_j^{(2)} &= \alpha x_j^{(1)} - \alpha x_j^{(2)}, \quad j = 1 \text{ or } 2.
\end{aligned} \tag{3.2}$$

- a system with two-first order delays:

$$\begin{aligned}
\dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)} + \gamma_2 x_2^{(1)}),\\
\dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2),\\
\dot{x}_1^{(1)} &= \alpha x_1 - \alpha x_1^{(1)},\\
\dot{x}_2^{(1)} &= \beta x_2 - \beta x_2^{(1)},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\dot{x}_1 &= x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)}),\\
\dot{x}_2 &= x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 x_2^{(1)}),
\end{aligned}$$

$$\dot{x}_1^{(1)} = \alpha x_1 - \alpha x_1^{(1)},$$
$$\dot{x}_2^{(1)} = \beta x_2 - \beta x_2^{(1)}, \tag{3.4}$$

$$\dot{x}_1 = x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_2^{(1)}),$$
$$\dot{x}_2 = x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 x_1^{(1)}),$$
$$\dot{x}_1^{(1)} = \alpha x_1 - \alpha x_1^{(1)},$$
$$\dot{x}_2^{(1)} = \beta x_2 - \beta x_2^{(1)} \tag{3.5}$$

$$\dot{x}_1 = x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma_1 x_1^{(1)}),$$
$$\dot{x}_2 = x_2(e_2 + a_{21}x_1 + a_{22}x_2 + \gamma_2 \tilde{x}_1^{(1)}),$$
$$\dot{x}_1^{(1)} = \alpha x_1 - \alpha x_1^{(1)},$$
$$\dot{\tilde{x}}_1^{(1)} = \beta x_1 - \beta \tilde{x}_1^{(1)}. \tag{3.6}$$

We will distinguish between two systems in (3.1) as $(3.1)_j$ for $j = 1, 2$. Similarly, we define system $(3.2)_j$ for $j=1,2$. For all systems, we always assume that $a_{ii} < 0$, $e_i \neq 0$ $(i = 1, 2)$ and $\alpha, \beta > 0$. The first assumptions imply self-crowding effects biologically and the last comes from (2.3).

First, we consider the boundedness of the solutions to systems $(3.1)_j$–(3.6). Note that $\mathbf{R}_+^3$ or $\mathbf{R}_+^4$ is positive invariant for each system.

Theorem 3.1. *Suppose that*
(a) *for* $(3.1)_1$; *one of the following is satisfied*:
- (a.1) $a_{12}a_{21} < 0$ *and* $a_{11} + \gamma < 0$,
- (a.2) $a_{12} \leqslant 0$, $a_{21} \leqslant 0$ *and* $a_{11} + \gamma < 0$,
- (a.3) $a_{11}a_{22} > a_{12}a_{21}$ *and* $\gamma < 0$,

(b) *for* $(3.1)_2$; *one of the following is satisfied*:
- (b.1) $a_{12}a_{21} < 0$ *and* $a_{11}a_{22} > -\gamma^2 a_{21}/(4a_{12})$,
- (b.2) $a_{12} \leqslant 0$ *and* $a_{21} \leqslant 0$,
- (b.3) $a_{11}a_{22} > a_{21}a_{21}$ *and* $\gamma \leqslant 0$,

(c) *for* $(3.2)_1$; *one of the following is satisfied*:
- (c.1) $a_{12}a_{21} < 0$ *and* $a_{11} + |\gamma| < 0$,
- (c.2) $a_{12} \leqslant 0$, $a_{21} \leqslant 0$ *and* $a_{11} + |\gamma| < 0$,
- (c.3) $a_{11}a_{22} > |a_{12}||a_{21}|$, $a_{11} + |a_{12}| < 0$ *and* $\gamma \leqslant 0$,

(d) *for* $(3.2)_2$; *one of the following is satisfied*:
- (d.1) $-a_{11} > |a_{12}| + |\gamma|$ *and* $-a_{22} > |a_{21}|$,
- (d.2) *the same as* (c.2),
- (d.3) *the same as* (c.3),

(e) *for* (3.3); *one of the following is satisfied*:
(e.1) $-a_{11} > |a_{12}| + |\gamma_1| + |\gamma_2|$ *and* $-a_{22} > |a_{21}|$,
(e.2) $a_{12} \leqslant 0$, $a_{21} \leqslant 0$ *and* $-a_{11} > |\gamma_1| + |\gamma_2|$,
(e.3) $a_{12}a_{22} > |a_{12}||a_{21}|$, $-a_{11} > |a_{12}|$, $\gamma_1 \leqslant 0$ *and* $\gamma_2 \leqslant 0$,

(f) *for* (3.4) *or* (3.5) *or* (3.6); *one of the following is satisfied*:
(f.1) $a_{12} \leqslant 0$, $a_{21} \leqslant 0$, $-a_{11} > |\gamma_1|$ *and* $-a_{22} > |\gamma_2|$,
(f.2) *the same as* (e.3).

Then solutions of $(3.1)_j$–(3.6) *are bounded for any* $\alpha > 0$ *and* $\beta > 0$.

Proof. Let us consider system $(3.1)_j$ and function

$$S_j = \sum_{i=1}^{2} \omega_i x_i + \frac{1}{2}\omega_3 \{x_j^{(1)}\}^2, \quad j = 1,2, \tag{3.7}$$

where $\omega_i > 0$ $(i = 1,2,3)$ are constants chosen suitably later. We consider only $(3.1)_1$ and denote S_1 simply by S. The time derivative of S along the solution of $(3.1)_1$ satisfies for any $\varepsilon > 0$ that

$$\dot{S} + \varepsilon S = \omega_1 x_1(e_1 + a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)}) + \omega_2 x_2(e_2 + a_{21}x_1 + a_{22}x_2) + \omega_3 x_1^{(1)}(\alpha x_1 - \alpha x_1^{(1)}) + \varepsilon \sum_{i=1}^{2} \omega_i x_i + \frac{1}{2}\omega_3 \varepsilon \{x_1^{(1)}\}^2.$$

We consider two cases.

Case 1: $a_{ij} \leqslant 0$ $(i,j = 1,2;\ i \neq j)$ or $a_{12}a_{21} < 0$. For the first case without any restriction on $\omega_i > 0$ $(i = 1,2,3)$ and for the second with $\omega_1 a_{12} + \omega_2 a_{21} = 0$, we have the following:

$$\dot{S} + \varepsilon S \leqslant \omega_1(e_1 + \varepsilon)x_1 + \omega_2(e_2 + \varepsilon)x_2 + \omega_2 a_{22} x_2^2 + (x_1,\ x_1^{(1)}) \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_3 \end{pmatrix} \begin{pmatrix} a_{11} & \gamma \\ \alpha & \varepsilon/2 - \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_1^{(1)} \end{pmatrix}. \tag{3.8}$$

Now we choose $\varepsilon > 0$ satisfying $\varepsilon/2 - \alpha < 0$ and $a_{11}(\varepsilon/2 - \alpha) > \alpha\gamma$, that is, satisfying

$$0 < \frac{\varepsilon}{2} < \min\left\{\alpha, \alpha + \frac{\alpha\gamma}{a_{11}}\right\}. \tag{3.9}$$

The choice of $\varepsilon > 0$ satisfying (3.9) is possible if $a_{11} + \gamma < 0$. Under this $\varepsilon > 0$, the matrix

$$\frac{1}{2}\left[\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_3 \end{pmatrix}\begin{pmatrix} a_{11} & \gamma \\ \alpha & \varepsilon/2 - \alpha \end{pmatrix} + \begin{pmatrix} a_{11} & \alpha \\ \gamma & \varepsilon/2 - \alpha \end{pmatrix}\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_3 \end{pmatrix}\right] \tag{3.10}$$

is negative definite under a suitable choice of $\omega_1,\ \omega_3 > 0$ (see for example [10]). Hence for any x_1 and $x_1^{(1)}$, the last term in (3.8) satisfies

$$(x_1,\ x_1^{(1)})\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_3\end{pmatrix}\begin{pmatrix}a_{11} & \gamma\\ \alpha & \varepsilon/2-\alpha\end{pmatrix}\begin{pmatrix}x_1\\ x_1^{(1)}\end{pmatrix} \leqslant -\lambda_M(x_1^2+\{x_1^{(1)}\}^2),$$

where $-\lambda_M$ is the largest eigenvalue of (3.10), which is negative. Therefore, we obtain

$$\dot{S}+\varepsilon S \leqslant x_1[-\lambda_M x_1+\omega_1(e_1+\varepsilon)]+\omega_2 x_2[a_{22}x_2+(e_2+\varepsilon)]-\lambda_M\{x_1^{(1)}\}^2,$$

which implies that $\dot{S}+\varepsilon S \leqslant M$ for any $x_1, x_2, x_1^{(1)}$, where M is some positive constant. This shows the boundedness of function S and solution of $(3.1)_1$ is bounded if one of (a.1) or (a.2) is satisfied.

Case 2: $\gamma < 0$. For this case, by choosing $\omega_1\gamma+\omega_3\alpha=0$, we have

$$\dot{S}+\varepsilon S=\omega_1(e_1+\varepsilon)x_1+\omega_2(e_2+\varepsilon)x_2+\omega_3(-\alpha+\varepsilon/2)\{x_1^{(1)}\}^2$$
$$+(x_1,\ x_2)\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix}\begin{pmatrix}a_{11} & a_{21}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

If $a_{11}a_{22} > a_{12}a_{21}$, the last term of the above is negative definite for any x_1, x_2 under the suitable choice of $\omega_1, \omega_2 > 0$. Further if we choose $\varepsilon > 0$ satisfying $-\alpha+\varepsilon/2 < 0$, by the same reason as Case 1 there exists a constant $M > 0$ such that $\dot{S}+\varepsilon S \leqslant M$ for any $x_1, x_2, x_1^{(1)}$. This shows the boundedness of the solution of $(3.1)_1$ under (a.3).

Similarly we can apply for $(3.2)_j$

$$S=\sum_{i=1}^{2}\omega_i x_i+\frac{1}{2}\omega_3\{x_j^{(1)}\}^2+\frac{1}{2}\omega_4\{x_j^{(2)}\}^2$$

for (3.3), (3.4), (3.5)

$$S=\sum_{i=1}^{2}\omega_i x_i+\frac{1}{2}\omega_3\{x_1^{(1)}\}^2+\frac{1}{2}\omega_4\{x_2^{(1)}\}^2$$

and for (3.6)

$$S=\sum_{i=1}^{2}\omega_i x_i+\frac{1}{2}\omega_3\{x_1^{(1)}\}^2+\frac{1}{2}\omega_4\{\tilde{x}_1^{(1)}\}^2,$$

respectively. The proof is just a repetition of the method for $(3.1)_1$ and we will omit it.

Remark 3.1. Let us consider for $(3.1)_1$ the relationship between the local stability of a positive equilibrium $E_+=(x_1^*, x_2^*, x_1^{(1)*})$ and the conditions on the boundedness of the solutions (that is, conditions (a.1)–(a.3)). The Jacobian of $(3.1)_1$ at E_+ is

$$J=\begin{pmatrix}a_{11}x_1^* & a_{12}x_1^* & \gamma x_1^*\\ a_{21}x_2^* & a_{22}x_2^* & 0\\ \alpha & 0 & -\alpha\end{pmatrix}.$$

Note that $x_1^{(1)*} = x_1^*$. The E_+ is stable if and only if

$$a_0 = -a_{11}x_1^* - a_{22}x_2^* + \alpha > 0,$$

$$a_1 = (a_{11}a_{22} - a_{12}a_{21})x_1^*x_2^* - \alpha(a_{11} + \gamma)x_1^* - \alpha a_{22}x_2^* > 0,$$

$$a_2 = (a_{11}a_{22} + a_{22}\gamma - a_{12}a_{21})\alpha x_1^*x_2^* > 0,$$

$$a_0a_1 - a_2 > 0.$$

It is easy to check that E_+ is stable if one of (a.1) and (a.3) is satisfied. The following examples show that (a.2) is not sufficient for E_+ to be stable.

Example 3.1. $a_{ii} = \gamma = -1$ $(i = 1, 2)$, $a_{12} = a_{21} = -2$, $e_1 = 4$ and $e_2 = 3$. For this case, (a.2) is satisfied and the solution of $(3.1)_1$ is bounded. Note that $E_+ = (1, 1, 1)$, $a_2 < 0$ for any $\alpha > 0$ and E_+ is unstable.

Example 3.2. $a_{ii} = -1$ $(i = 1, 2)$, $\gamma = 0.7$, $a_{12} = -1$, $a_{21} = -0.5$, $e_1 = 1.3$ and $e_2 = 1.5$. Since (a.2) is satisfied, the solution of $(3.1)_1$ is bounded. Note that again $E_+ = (1, 1, 1)$, $a_2 < 0$ for any $\alpha > 0$ and E_+ is unstable. Further note that

$$-A = \begin{pmatrix} 1 & 1 & -0.7 \\ 0.5 & 1 & 0 \\ -\alpha & 0 & \alpha \end{pmatrix} \notin S_w \quad \text{for any } \alpha > 0,$$

since $\det(-A) = -0.2\alpha < 0$ [10]. Hence condition (a.2) extends the condition on the boundedness of the solution given in Theorem 2.1.

Now let us show "partial permanence" of the solutions to $(3.1)_j$–(3.6). We need the following Butler–McGehee's lemma [4].

Butler–McGehee's (B–M) lemma. *Suppose that P is a hyperbolic equilibrium point of*

$$\dot{y} = f(y) \quad (y \in \mathbf{R}^n, f\colon \mathbf{R}^n \to \mathbf{R}^n, f \in C^1)$$

which is in $\omega(x)$ (the omega limit set of a positive orbit), but is not the entire omega limit set. Then $\omega(x)$ has nontrivial (i.e., different from P) intersection with the stable manifold $M^+(P)$ and the unstable manifold $M^-(P)$ of P.

Remark 3.2. From [8], we know that the Jacobian matrix at such a point P satisfying B–M lemma cannot have all of its eigenvalues with negative real part; it also is impossible to have all eigenvalues with positive real part. Hence the stable and unstable manifolds are not empty. The lemma implies that an orbit cannot sneak into and out of a neighborhood of P infinitely often without having accumulation points on $M^+(P)$ and $M^-(P)$, when $P \in \omega(x)$ but $P \neq \omega(x)$.

We have the following results:

Theorem 3.2. *Suppose that the solutions of $(3.1)_j$–(3.6) are bounded and at least one of e_i $(i = 1, 2)$ is positive. Consider the solution $x(t)$ starting in $\mathbf{R}^3_+$ (system*

$(3.1)_j$) *or in* $\mathbf{R}^4_+$ (*system* $(3.2)_j$–(3.6)). *Choose a sufficiently large number* $T > 0$ *and a sufficiently small number* $\varepsilon > 0$ *and define sets*

$$\Omega^3_j = \{x \in R^3_+ | x_1 + x_2 > \varepsilon, x^{(1)}_j > 0\}, \quad j = 1, 2,$$

$$\Omega^4 = \{x \in R^4_+ | x_1 + x_2 > \varepsilon, x^{(1)}_j > 0,\ j = 1, 2\},$$

$$\bar{\Omega}^4 = \{x \in R^4_+ | x_i > \varepsilon, x^{(i)}_1 > 0,\ i = 1, 2\},$$

$$\tilde{\Omega}^4 = \{x \in R^4_+ | x_1 + x_2 > \varepsilon, x^{(1)}_1 > 0,\ \tilde{x}^{(1)}_1 > 0\}.$$

(i) *For* $(3.1)_1$, *the solution stays in* Ω^3_1 *for* $t > T$, *if* $\gamma \leqslant 0$ *or* $-a_{11} > \gamma > 0$.
(ii) *For* $(3.1)_2$, *the solution stays in* Ω^3_2 *for* $t > T$.
(iii) *Suppose that* $-a_{11} > |\gamma|$. *Then for* $(3.2)_1$, *the solution stays in* $\bar{\Omega}^4$ *for* $t > T$, *if*

$$e_2 > a_{21}e_1/(a_{11} + \gamma) \quad \textit{when } e_1 > 0,$$

$$\text{or} \quad e_1 > a_{12}e_2/a_{22} \quad \textit{when } e_2 > 0. \tag{3.11}$$

(iv) *For* $(3.2)_2$, *the solution stays in* $\bar{\Omega}^4$ *for* $t > T$, *if*

$$e_2 > a_{21}e_1/a_{11} \quad \textit{when } e_1 > 0,$$

$$\text{or} \quad e_1 > e_2(a_{12} + \gamma)/a_{22} \quad \textit{when } e_2 > 0. \tag{3.12}$$

(v) *For* (3.3), *the solution stays in* Ω^4 *for* $t > T$, *if* $-a_{11} > |\gamma_1|$.
(vi) *For* (3.4), *the solution stays in* Ω^4 *for* $t > T$, *if*

$$-a_{ii} > |\gamma_i| \quad (i = 1, 2). \tag{3.13}$$

(vii) *For* (3.5), *the solution stays in* Ω^4 *for* $t > T$.
(viii) *For* (3.6), *the solution stays in* $\tilde{\Omega}^4$ *for* $t > T$, *if* $-a_{11} > |\gamma_1|$.

Proof. Consider system $(3.1)_1$. First we show that the equilibrium point $E_0 = (0, 0, 0)$ of $(3.1)_1$ is not contained in the ω-limit set of any solution of $(3.1)_1$ starting at a point belonging to $\mathbf{R}^3_+$.

Let $\gamma^+(x)$ be the positive orbit through a point $x \in \mathbf{R}^3_+$ of $(3.1)_1$ and $\omega(x)$ be its ω-limit set. Note that $\omega(x)$ is not empty since the solution is assumed to be bounded. Note that the Jacobian matrix of $(3.1)_1$ at E_0 is

$$J_0 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix}$$

and the E_0 is a saddle since at least one of e_i $(i = 1, 2)$ is assumed to be positive and $\alpha > 0$. Hence $\omega(x) \neq E_0$ by Remark 3.2.

Now let us show that $E_0 \notin \omega(x)$.

Case 1: $e_1 > 0$, $e_2 > 0$ (see Fig. 1(a)). Assume that $E_0 \in \omega(x)$. Since $E_0 \neq \omega(x)$, by B–M lemma, there is at least one point $Q \in \omega(x) \cap M^+(E_0)$, $Q \neq E_0$. Note that $M^+(E_0)$ is the positive $x^{(1)}_1$-axis by $e_i > 0$ $(i = 1, 2)$. Since the entire orbit through Q

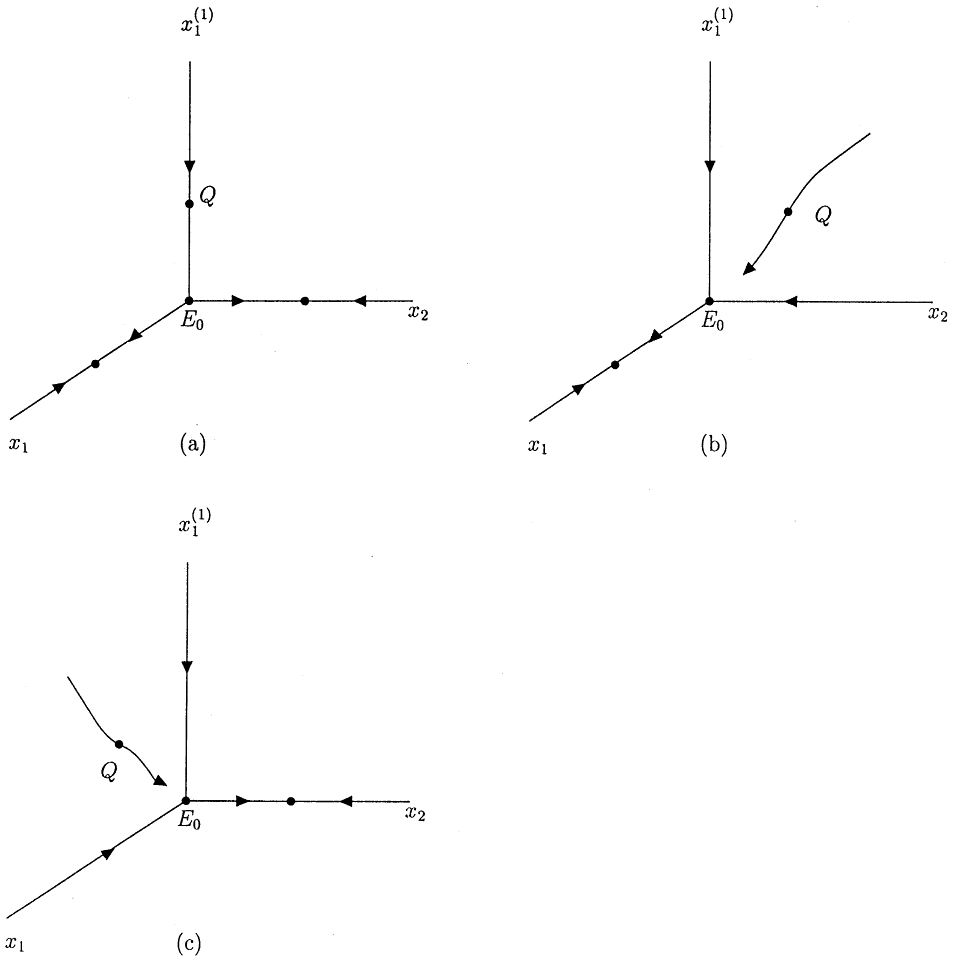

Fig. 1. By B–M lemma, there exists a point $Q \in \omega(x) \cap M^+(E_0)$ when $E_0 \in \omega(x)$ but $E_0 \neq \omega(x)$. (a) $M^+(E_0) = \{(x_1, x_2, x_1^{(1)} | x_1 = x_2 = 0, x_1^{(1)} > 0\}$. (b) $M^+(E_0) = \{(x_1, x_2, x_1^{(1)} | x_1 = 0,\ x_2 \geqslant 0,\ x_1^{(1)} \geqslant 0\}$. (c) $M^+(E_0) = \{(x_1, x_2, x_1^{(1)} | x_1 \geqslant 0,\ x_2 = 0,\ x_1^{(1)} \geqslant 0\}$.

is contained in $\omega(x)$, the positive $x_1^{(1)}$-axis is contained in $\omega(x)$, which contradicts to the boundedness of the solution. Hence $E_0 \notin \omega(x)$.

Case 2: $e_1 > 0$, $e_2 < 0$ (see Fig. 1(b)). Assume that $E_0 \in \omega(x)$. Again by B–M lemma, there exists a point $Q \in \omega(x) \cap M^+(E_0)$ and $Q \neq E_0$. The $M^+(E_0)$ is the nonnegative x_2–$x_1^{(1)}$ plane. It is trivial that the plane is invariant for $(3.1)_1$. The system on the plane is described by

$$\dot{x}_2 = x_2(e_2 + a_{22}x_2), \quad \dot{x}_1^{(1)} = -\alpha x_1^{(1)}$$

and any solution starting at a point in the plane tends to $(0,0)$ as $t \to +\infty$. Note that $\dot{x}_2 \leqslant 0$ since $e_2 < 0$, $a_{22} < 0$. If Q is on either boundary axis of the plane, we have the similar contradiction to the boundedness as Case 1. If Q belongs to the interior of the plane, since there is no equilibrium and no periodic orbit in the interior of the plane, the orbit through Q must be unbounded, giving a contradiction. Hence, $E_0 \notin \omega(x)$.

Case 3: $e_1 < 0$, $e_2 > 0$ (see Fig. 1(c)). Suppose again that $E_0 \in \omega(x)$. By B–M lemma, there exists a point $Q \in \omega(x) \cap M^+(E_0)$ and $Q \neq E_0$. Now $M^+(E_0)$ is the nonnegative x_1–$x_1^{(1)}$ plane. The system on the plane is

$$\dot{x}_1 = x_1(e_1 + a_{11}x_1 + \gamma x_1^{(1)}), \quad \dot{x}_1^{(1)} = \alpha x_1 - \alpha x_1^{(1)}.$$

It is easy to check that there exists neither equilibrium point nor periodic orbit in the interior of the plane under the condition that $\gamma \leqslant 0$ or $-a_{11} > \gamma > 0$. Hence any solution starting at a point in the plane tends to $(0,0)$ as $t \to \infty$. This gives the similar contradiction to the boundedness of the solution as Case 2. This shows that $E_0 \notin \omega(x)$ for all cases.

Now suppose that for some $x > 0$, the $\omega(x)$ contains a point on the positive $x_1^{(1)}$-axis. Since the axis is positively invariant, this assumption implies that $\omega(x)$ contains E_0 or is unbounded, both give contradictions. Hence, $\omega(x)$ contains no point of the $x_1^{(1)}$-axis. This shows that there exists no sequence $\{t_n\}$ of real numbers, which tends to infinity as $n \to \infty$ such that $x_1(t_n) \to 0$, $x_2(t_n) \to 0$, $x_1^{(1)} \to \bar{x}_1^{(1)} \geqslant 0$ as $n \to \infty$, for any $\bar{x}_1^{(1)} \geqslant 0$. This proves (i).

Consider now system $(3.1)_2$. Note that the dynamics in the nonnegative x_1–$x_2^{(1)}$ plane of $(3.1)_2$ is described as

$$\dot{x}_1 = x_1(e_1 + a_{11}x_1 + \gamma x_2^{(1)}), \quad \dot{x}_2^{(1)} = -\alpha x_2^{(1)}.$$

Hence any solution starting at a point in the plane tends to $(0,0)$ as $t \to \infty$ when $e_1 < 0$ and $e_2 > 0$. This is a qualitatively different point from Case 3 for $(3.1)_1$ and now we have no restriction on $a_{11} < 0$ and γ. This shows (ii).

Consider system $(3.2)_1$. We will show under assumption (iii) that boundary equilibrium points $E_0 = (0,0,0,0)$, $E_1^{12} = (x_1^*, 0, x_1^*, x_1^*)$ and $E_2 = (0, -e_2/a_{22}, 0, 0)$, where $x_1^* = -e_1/(a_{11} + \gamma) > 0$, are not contained in the ω-limit set of any solution of $(3.2)_1$ starting at $\mathbf{R}_+^4$.

First we prove the following:

(P1) when E_2 exists (that is, when $e_2 > 0$), it is globally asymptotically stable with respect to $\mathbf{R}_2 = \{x \in \mathbf{R}_{+0}^4 | x_1 = 0, x_2 > 0,\ x_1^{(1)} \geqslant 0,\ x_1^{(2)} \geqslant 0\}$.

(P2) When E_2 does not exist, E_0 is globally asymptotically stable with respect to $\{x \in \mathbf{R}_{+0}^4 | x_1 = 0, x_2 \geqslant 0, x_1^{(1)} \geqslant 0, x_1^{(2)} \geqslant 0\}$.

(P3) When E_1^{12} exists (that is, when $e_1 > 0$), it is globally asymptotically stable with respect to the nonnegative $x_1 - x_1^{(1)} - x_1^{(2)}$ space $\mathbf{R}_1 = \{x \in \mathbf{R}_{+0}^4 | x_1 > 0, x_2 = 0, x_1^{(1)} > 0, x_1^{(2)} > 0\}$.

(P4) When E_1^{12} does not exist, E_0 is globally asymptotically stable with respect to $\{x \in \mathbf{R}_{+0}^4 | x_1 \geqslant 0, x_2 = 0, x_1^{(1)} \geqslant 0, x_1^{(2)} \geqslant 0\}$.

In fact, the space $\mathbf{R}_2$ is positively invariant and system $(3.2)_1$ on the space is described as

$$\dot{x}_2 = x_2(e_2 + a_{22}x_2), \quad \dot{x}_1^{(1)} = -\alpha x_1^{(1)}, \quad \dot{x}_1^{(2)} = \alpha x_1^{(1)} - \alpha x_1^{(2)},$$

which shows that $x_2 \to -e_2/a_{22}$, $x_1^{(j)} \to 0$ $(j = 1,2)$ as $t \to +\infty$ when E_2 exists (and $e_2 > 0$). This shows (P1). It is easy to show (P2) when $e_2 \leqslant 0$.

Condition (P3) follows from the results of [3]. Suppose that E_1^{12} does not exist $(e_1 < 0)$. Since the space $\{x \in \mathbf{R}^4_{+0} | x_1 \geqslant 0, x_2 = 0, x_1^{(1)} \geqslant 0, x_1^{(2)} \geqslant 0\}$ is positively invariant and $(3.2)_1$ on the space is

$$\dot{x}_1 = x_1(e_1 + a_{11}x_1 + \gamma x_1^{(2)}), \quad \dot{x}_1^{(1)} = \alpha x_1 - \alpha x_1^{(1)}, \quad \dot{x}_1^{(2)} = \alpha x_1^{(1)} - \alpha x_1^{(2)}.$$

Consider the function $V = \ln x_1 + \omega(x_1^{(1)} + x_1^{(2)})$ for some $\omega > 0$. Then from $-a_{11} > \gamma$, we have

$$\dot{V} \leqslant e_1 + (\omega\alpha + a_{11})(x_1 - x_1^{(2)}).$$

By choosing $\omega\alpha + a_{11} = 0$, we have $V(t) \leqslant V(0) + e_1 t$ $(t \geqslant 0)$, which shows that $x_1(t) \to 0$ as $t \to \infty$ (note that $e_1 < 0$). This shows (P4).

Now we are in position to prove (iii). First we show that $E_0 \notin \omega(x)$. Assume that $E_0 \in \omega(x)$. The $M^+(E_0)$ is the positive $x_1^{(1)} - x_1^{(2)}$ plane when $e_1 > 0$ and $e_2 > 0$ (Case 1: see Fig. 2(a)). By the Jacobian matrix at E_0, we have $E_0 \neq \omega(x)$ by Remark 3.2 and there exists at least one point $Q_0 \in \omega(x) \cap M^+(E_0)$, $Q_0 \neq E_0$ by B–M lemma. It is easy to check that the $M^+(E_0)$ is invariant and any solution starting at a point on $M^+(E_0)$ tends to $(0,0)$ as $t \to \infty$. Since there is no equilibrium point and no periodic orbit on $M^+(E_0)$, the orbit through Q_0 must be unbounded, giving a contradiction. Now consider the case where $e_1 < 0$ and $e_2 > 0$ (Case 2: see Fig. 2(b)). This case $M^+(E_0)$ is contained in the positive x_1–$x_1^{(1)}$–$x_1^{(2)}$ space. If $E_0 \in \omega(x)$, then there exists a Q_0 such that $Q_0 \in \omega(x) \cap M^+(E_0)$. Since $e_1 < 0$, by (P4), E_0 is globally asymptotically stable with respect to the nonnegative x_1–$x_1^{(1)}$–$x_1^{(2)}$ space. By the same reason for Case 1, the orbit through Q_0 must be unbounded, giving a contradiction. When $e_1 > 0$ and $e_2 < 0$ (Case 3: see Fig. 2(c)), $E_0 \in \omega(x)$ gives the same contradiction, by (P2). This shows that $E_0 \notin \omega(x)$ for all cases.

Next we show that $E_2 \notin \omega(x)$ and $E_1^{12} \notin \omega(x)$. We consider only Case 1, the remaining Case 2 for E_2 and Case 3 for E_1^{12} are proved similarly. Suppose first that $E_2 \in \omega(x)$. From the Jacobian matrix at E_2 and assumption (iii) (i.e. $e_1 > a_{12}e_2/a_{22}$ when $e_2 > 0$), $M^+(E_2)$ is contained in the set $\mathbf{R}_2$. By B–M lemma, there exists a point $Q_2 \in \omega(x) \cap M^+(E_2)$. If Q_2 is contained in the positive $x_1^{(1)}$–$x_1^{(2)}$ plane $(x_2 = 0)$, we have the same contradiction as one we have for E_0. If Q_2 is a point on the positive x_2-axis, we have two possibilities; one has $E_0 \in \omega(x)$ or an unbounded solution; both give a contradiction (note that the positive x_2-axis is invariant and has a globally asymptotically stable point $-e_2/a_{22}$ with respect to itself). If $Q_2 \in \operatorname{int} \mathrm{R}_2$, again we have two possibilities by (P1); one has $E_0 \in \omega(x)$ and the other $\omega(x)$ is unbounded, both give a contradiction. This shows that $E_2 \notin \omega(x)$.

Finally let us assume that $E_1^{12} \in \omega(x)$. From the Jacobian matrix at E_1^{12} and assumption (iii) (i.e., $e_2 > a_{21}e_1/(a_{11} + \gamma)$ when $e_1 > 0$), $M^+(E_1^{12})$ is contained in $\mathbf{R}_1$. By B–M lemma, there exists a point $Q_1 \in \omega(x) \cap M^+(E_1^{12})$. Again by (P3), we have two possibilities; one has $E_0 \in \omega(x)$ and the other $\omega(x)$ is unbounded, both a contradiction. This completes the proof that none of E_0, E_1^{12} or E_2 are not contained in $\omega(x)$ of $(3.2)_1$.

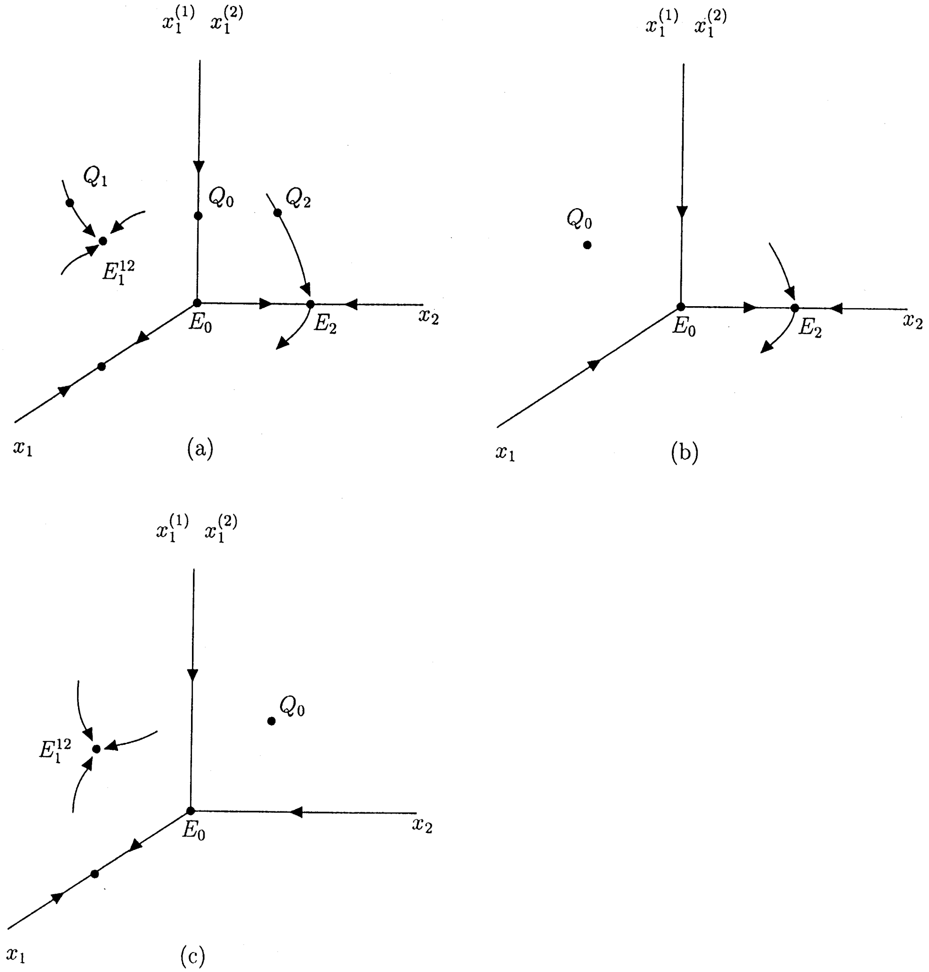

Fig. 2. By B–M lemma, there exists a point $Q_0 \in \omega(x) \cap M^+(E_0)$ when $E_0 \in \omega(x)$ but $E_0 \neq \omega(x)$. (a) $M^+(E_0)$ is the positive $x_1^{(1)}$–$x_1^{(2)}$ plane. (b) $M^+(E_0)$ is the positive $x_1 - x_1^{(1)}$–$x_1^{(2)}$ space. (c) $M^+(E_0)$ is contained in the positive x_1–$x_1^{(1)}$–$x_1^{(2)}$ space.

The above also shows that any point in spaces $\mathbf{R}_1, \mathbf{R}_2$ and in the nonnegative $x_1^{(1)}$–$x_1^{(2)}$ plane is not contained in $\omega(x)$ of any solution of $(3.2)_1$. This implies that any solution of $(3.2)_1$ starting in $\mathbf{R}_+^4$ stays in $\bar{\Omega}_4$ for $t > T$ and completes the proof of (iii).

For $(3.2)_2$, we can prove similarly that the boundary equilibrium points E_0, $E_1 = (-e_1/a_{11}, 0, 0, 0)$, $E_2^{12} = (0, -e_2/a_{22}, -e_2/a_{22}, -e_2/a_{22})$ are not contained in $\omega(x)$ under the condition given in (iv) and it follows just the same as (iii). For the remaining systems (3.3)–(3.6), we can prove that $E_0 \notin \omega(x)$ and (v)–(viii) follow similarly as (i). This completes the proof of Theorem 3.2.

4. Nonexistence of periodic solutions

Let us apply Li–Muldowney's criteria (Corollary 2.8) for the nonexistence of periodic solutions of systems $(3.1)_j$–(3.6) ($j = 1,2$). Choose A as the identity matrix in (2.13) and consider $(3.1)_1$. The Jacobian matrix of $(3.1)_1$ becomes

$$J = \begin{pmatrix} e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)} & a_{12}x_1 & \gamma x_1 \\ a_{21}x_2 & e_2 + a_{21}x_1 + 2a_{22}x_2 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix}.$$

The logarithmic norm μ_1 endowed by the norm $|x|_1$ of the second additive compound matrix $J^{[2]}$ associated to J is negative in $\mathbf{R}^3_{+0}$ if and only if the supremums of the following functions satisfy

$$(e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)}) + (e_2 + a_{21}x_1 + 2a_{22}x_2) + \alpha < 0,$$

$$(e_1 + 2a_{11}x_1 + a_{12}x_2 + \gamma x_1^{(1)}) - \alpha + |a_{21}|x_2 < 0,$$

$$(e_2 + a_{21}x_1 + 2a_{22}x_2) - \alpha + |a_{12}|x_1 + |\gamma|x_1 < 0,$$

in $\mathbf{R}^3_{+0}$. From the second and third inequalities, we have $a_{12} + |a_{21}| \leqslant 0$ and $a_{21} + |a_{12}| + |\gamma| \leqslant 0$ as necessary conditions for $\mu_1 < 0$ in $\mathbf{R}^3_{+0}$. These two conditions hold true only for $\gamma = 0$, which gives us a Lotka–Volterra system without a delay term. This shows that the *direct* application of Li–Muldowney's method does not work for $(3.1)_1$.

Now let us transform $(3.1)_1$ by change of variables

$$\dot{y}_1 = (e_1 + a_{11}e^{\lambda_1 y_1} + a_{12}e^{\lambda_2 y_2} + \gamma x_1^{(1)})/\lambda_1,$$

$$\dot{y}_2 = (e_2 + a_{21}e^{\lambda_1 y_1} + a_{22}e^{\lambda_2 y_2})/\lambda_2,$$

$$\dot{x}_1^{(1)} = \alpha e^{\lambda_1 y_1} - \alpha x_1^{(1)}, \tag{4.1}$$

where new variables y_i $(i = 1,2)$ are defined by $y_i = (\log x_i)/\lambda_i$, for some positive constants λ_i chosen later. The Jacobian matrix of (4.1) is

$$J_1^1 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 \\ \alpha\lambda_1 e^{\lambda_1 y_1} & 0 & -\alpha \end{pmatrix}.$$

The logarithmic norm $\mu_1(J_1^{1^{[2]}})$ is negative in $\mathbf{R}^3$ (note that it must be negative in $\mathbf{R}^3$, not in $\mathbf{R}^3_{+0}$, because of change of variables) if and only if the following is satisfied in $\mathbf{R}^3$

$$\sup\{a_{11}e^{\lambda_1 y_1} + a_{22}e^{\lambda_2 y_2} + \alpha\lambda_1 e^{\lambda_1 y_1}\} < 0,$$

$$\sup\{a_{11}e^{\lambda_1 y_1} - \alpha + \lambda_1 |a_{21}| e^{\lambda_1 y_1}/\lambda_2\} < 0,$$

$$\sup\{a_{22}e^{\lambda_2 y_2} - \alpha + \lambda_2 |a_{12}| e^{\lambda_2 y_2}/\lambda_1 + |\gamma|/\lambda_1\} < 0. \tag{4.2}$$

Suppose that for sufficiently small $\varepsilon > 0$ and large $T > 0$, the following is satisfied by the solution $y(t) = (y_1(t), y_2(t), x_1^{(1)}(t))$ of (4.1)

$$y(t) \in \Omega_{1y}^3 = \{y \in R^3 | e^{\lambda_1 y_1} + e^{\lambda_2 y_2} > \varepsilon, x_1^{(1)} > 0\} \quad \text{for } t > T. \tag{4.3}$$

Under assumption (4.3), the conditon given in Corollary 2.8 is ensured if

$$a_{11} + \alpha\lambda_1 < 0, \quad a_{11} + \lambda_1|a_{21}|/\lambda_2 \leqslant 0,$$

$$a_{22} + \lambda_2|a_{12}|/\lambda_1 \leqslant 0, \quad -\alpha + |\gamma|/\lambda_1 < 0.$$

The above is equivalent to

$$-\frac{|a_{21}|}{a_{11}} \leqslant \frac{\lambda_2}{\lambda_1} \leqslant -\frac{a_{22}}{|a_{12}|}, \quad \frac{|\gamma|}{\lambda_1} < \alpha < -\frac{a_{11}}{\lambda_1}. \tag{4.4}$$

Suppose that $a_{11}a_{22} \geqslant |a_{12}||a_{21}|$ and $-a_{11} > |\gamma|$. Then it is easy to check that we can choose $\lambda_i > 0$ $(i = 1, 2)$ satisfying (4.4) for each $\alpha > 0$. Note that $\Omega_1^{3_y}$ corresponds to Ω_1^3 defined in Section 3 and (4.3) is equivalent that the solution of $(3.1)_1$ stays in Ω_1^3 for $t > T$. For the last property, a sufficient condition is given in Theorem 3.2(i). This proves the following Theorem 4.1(i):

Theorem 4.1. *Suppose that the solutions of* $(3.1)_j$–(3.6) *are bounded and at least one of* e_i $(i = 1, 2)$ *is positive. Then each system has no periodic solutions for any* $\alpha > 0$ and $\beta > 0$ *if the following conditions are satisfied*:

(i) *For* $(3.1)_1$,

$$a_{11}a_{22} \geqslant |a_{12}||a_{21}|, \quad -a_{11} > |\gamma|. \tag{4.5}$$

(ii) *For* $(3.1)_2$,

$$a_{11}a_{22} \geqslant |a_{12}||a_{21}|, \quad a_{11}a_{22} > |a_{21}||\gamma|. \tag{4.6}$$

(iii) *For* $(3.2)_1$, (3.11) *and*

$$a_{22}(|\gamma| + a_{11}) > |a_{12}||a_{21}|. \tag{4.7}$$

(iv) *For* $(3.2)_2$, (3.12) *and*

$$a_{11}a_{22} > |a_{21}|(|\gamma| + |a_{12}|). \tag{4.8}$$

(v) *For* (3.3),

$$a_{22}(|\gamma_1| + a_{11}) > |a_{21}|(|\gamma_2| + |a_{12}|). \tag{4.9}$$

(vi) *For* (3.4), (3.13) *and*

$$(a_{11} + |\gamma_1|)(a_{22} + |\gamma_2|) > |a_{12}||a_{21}|. \tag{4.10}$$

(vii) *For* (3.5),

$$a_{11}a_{22} > (|\gamma_1| + |a_{12}|)(|\gamma_2| + |a_{21}|). \tag{4.11}$$

(viii) *For* (3.6),

$$a_{22}(|\gamma_1| + a_{11}) > |a_{12}||a_{21}|, \quad a_{11}a_{22} > |a_{12}|(|a_{21}| + |\gamma_2|) \tag{4.12}$$

and

$$|a_{21}| > |\gamma_2|; \tag{4.13}$$

or (4.12) *and*

$$-a_{11}|a_{21}| > (|a_{21}| + |\gamma_2|)|\gamma_1|, \quad 2|\gamma_1| + a_{11} < 0. \tag{4.14}$$

Proof. (ii) By the same change of variables as for $(3.1)_1$, the Jacobian matrix of the system corresponding to $(3.1)_2$ becomes

$$J_2^1 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 \\ 0 & \alpha\lambda_2 e^{\lambda_2 y_2} & -\alpha \end{pmatrix}.$$

By Theorem 3.2(ii), the solution stays in Ω_2^3 for $t > T$ and the solution $y(t) = (y_1(t), y_2(t), x_2^{(1)}(t))$ satisfies $y(t) \in \Omega_{2y}^3 = \{y \in \mathbf{R}^3 | e^{\lambda_1 y_1} + e^{\lambda_2 y_2} > \varepsilon, x_2^{(1)} > 0\}$ for $t > T$. By using this and J_2^1, it is easy to check that the logarithmic norm $\mu_1(J_2^{1[2]})$ is negative in Ω_{2y}^3 if

$$-\frac{|a_{21}|}{a_{11}} \leqslant \frac{\lambda_2}{\lambda_1} \leqslant -\frac{a_{22}}{|a_{12}|}, \quad \frac{|\gamma|}{\lambda_1} < \alpha < -\frac{a_{22}}{\lambda_2}. \tag{4.15}$$

We can choose $\lambda_i > 0$ $(i = 1, 2)$ satisfying the above for each $\alpha > 0$ under assumption (4.6). In fact, by (4.6), it is possible to choose λ_2/λ_1 satisfying

$$-\frac{|a_{21}|}{a_{11}} \leqslant \frac{\lambda_2}{\lambda_1} \leqslant \min\left\{-\frac{a_{22}}{|a_{12}|}, -\frac{a_{22}}{|\gamma|}\right\}.$$

Denote such a value of λ_2/λ_1 as k and the second inequality of (4.15) becomes $|\gamma|/\lambda_1 < \alpha < -a_{22}/(k\lambda_1)$. By changing the value of λ_1 from $0+$ to $+\infty$, the α can take any positive number. This shows (ii).

(iii) Similarly, we have for $(3.2)_1$, the Jacobian matrix

$$J_1^2 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & 0 & \gamma/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 & 0 \\ \alpha\lambda_1 e^{\lambda_1 y_1} & 0 & -\alpha & 0 \\ 0 & 0 & \alpha & -\alpha \end{pmatrix}.$$

By Theorem 3.2(iii) and (3.11), the solution $y(t)$ satisfies $y(t) \in \bar{\Omega}_y^4 = \{y \in \mathbf{R}^4 | e^{\lambda_i y_i} > \varepsilon, x_1^{(i)} > 0, (i = 1, 2)\}$ for $t > T$ (where T is sufficiently large). The logarithmic norm $\mu_1(J_1^{2[2]})$ is negative in $\bar{\Omega}_y^4$ if

$$\sup\{a_{11}e^{\lambda_1 y_1} + a_{22}e^{\lambda_2 y_2} + \alpha\lambda_1 e^{\lambda_1 y_1}\} < 0,$$

$$\sup\{a_{11}e^{\lambda_1 y_1} - \alpha + \lambda_1|a_{21}|e^{\lambda_1 y_1}/\lambda_2 + \alpha\} < 0,$$

$$\sup\{a_{11}e^{\lambda_1 y_1} - \alpha + \lambda_1|a_{21}|e^{\lambda_1 y_1}/\lambda_2 + \alpha\lambda_1 e^{\lambda_1 y_1}\} < 0,$$

$$\sup\{a_{22}e^{\lambda_2 y_2} - \alpha + \alpha + \lambda_2|a_{12}|e^{\lambda_2 y_2}/\lambda_1\} < 0,$$

$$\sup\{a_{22}e^{\lambda_2 y_2} - \alpha + \lambda_2|a_{12}|e^{\lambda_2 y_2}/\lambda_1 + |\gamma|/\lambda_1\} < 0,$$

$$\sup\{-\alpha - \alpha + |\gamma|/\lambda_1\} < 0.$$

Owing to $\bar{\Omega}^4_y$, (now we have each $e^{\lambda_i y_i} > \varepsilon$, not $e^{\lambda_1 y_1} + e^{\lambda_2 y_2} > \varepsilon$ as in Ω^3_{2y}), the above inequalities are satisfied if

$$\frac{\lambda_2}{\lambda_1} < -\frac{a_{22}}{|a_{12}|}, \quad \frac{|\gamma|}{\lambda_1} < \alpha < -\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}.$$

It is easy to check that we can choose $\lambda_i > 0$ $(i = 1, 2)$ satisfying the above for each α under (4.7). Note that (4.7) is equivalent to

$$|\gamma| + a_{11} < 0, \quad -|a_{21}|/(|\gamma| + a_{11}) < -a_{22}/|a_{12}|.$$

The remaining (iv)–(viii) can be proved similarly and we just give each Jacobian matrix and the condition for its logarithmic norm to be negative.

For $(3.2)_2$,

$$J^2_2 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & 0 & \gamma/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 & 0 \\ 0 & \alpha\lambda_2 e^{\lambda_2 y_2} & -\alpha & 0 \\ 0 & 0 & \alpha & -\alpha \end{pmatrix},$$

$$\frac{\lambda_2}{\lambda_1} > -\frac{|a_{21}|}{a_{11}}, \quad \frac{|\gamma|}{\lambda_1} < \alpha < -\frac{a_{22}}{\lambda_2} - \frac{|a_{12}|}{\lambda_1}.$$

For (3.3),

$$J^3 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma_1\lambda_3/\lambda_1 & \gamma_2/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 & 0 \\ \lambda_1\alpha e^{\lambda_1 y_1}/\lambda_3 & 0 & -\alpha & 0 \\ 0 & \beta\lambda_2 e^{\lambda_2 y_2} & 0 & -\beta \end{pmatrix},$$

$$\frac{|\gamma_1|}{\lambda_1}\lambda_3 < \alpha < \left(-\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}\right)\lambda_3, \quad \frac{|\gamma_2|}{\lambda_1} < \beta < \left(-\frac{a_{22}}{\lambda_2} - \frac{|a_{12}|}{\lambda_1}\right).$$

For (3.4),

$$J^4 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma_1\lambda_3/\lambda_1 & 0 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 & \gamma_2/\lambda_2 \\ \lambda_1\alpha e^{\lambda_1 y_1}/\lambda_3 & 0 & -\alpha & 0 \\ 0 & \beta\lambda_2 e^{\lambda_2 y_2} & 0 & -\beta \end{pmatrix},$$

$$\frac{|\gamma_1|}{\lambda_1}\lambda_3 < \alpha < \left(-\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}\right)\lambda_3, \quad \frac{|\gamma_2|}{\lambda_2} < \beta < \left(-\frac{a_{22}}{\lambda_2} - \frac{|a_{12}|}{\lambda_1}\right).$$

For (3.5),

$$J^5 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & 0 & \gamma_1/\lambda_1 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & \gamma_2\lambda_3/\lambda_2 & 0 \\ \lambda_1 \alpha e^{\lambda_1 y_1}/\lambda_3 & 0 & -\alpha & 0 \\ 0 & \beta\lambda_2 e^{\lambda_2 y_2} & 0 & -\beta \end{pmatrix},$$

$$\frac{|\gamma_2|}{\lambda_2}\lambda_3 < \alpha < \left(-\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}\right)\lambda_3, \quad \frac{|\gamma_1|}{\lambda_1} < \beta < \left(-\frac{a_{22}}{\lambda_2} - \frac{|a_{12}|}{\lambda_1}\right).$$

For (3.6),

$$J^6 = \begin{pmatrix} a_{11}e^{\lambda_1 y_1} & \lambda_2 a_{12}e^{\lambda_2 y_2}/\lambda_1 & \gamma_1\lambda_3/\lambda_1 & 0 \\ \lambda_1 a_{21}e^{\lambda_1 y_1}/\lambda_2 & a_{22}e^{\lambda_2 y_2} & 0 & \gamma_2/\lambda_2 \\ \lambda_1 \alpha e^{\lambda_1 y_1}/\lambda_3 & 0 & -\alpha & 0 \\ \beta\lambda_1 e^{\lambda_1 y_1} & 0 & 0 & -\beta \end{pmatrix},$$

$$\frac{\alpha}{\lambda_3} + \beta < -\frac{a_{11}}{\lambda_1}, \quad \frac{|\gamma_1|}{\lambda_1}\lambda_3 < \alpha < \left(-\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}\right)\lambda_3,$$

$$\frac{|\gamma_2|}{\lambda_2} < \beta < -\frac{a_{11}}{\lambda_1} - \frac{|a_{21}|}{\lambda_2}, \quad \frac{\lambda_2}{\lambda_1} < -\frac{a_{22}}{|a_{12}|}.$$

Here we chose new variables $y_i = (\ln x_i)/\lambda_i$ $(i = 1,2)$, $y_3 = x_1^{(1)}/\lambda_3$ for systems (3.3)–(3.6).

5. Discussion

Let us consider Example 3.2 again. The parameters satisfy (a.2) in Theorem 3.1 and (4.5) in Theorem 4.1. Hence, for system $(3.1)_1$ with these parameters, the solutions are bounded by Theorem 3.1 and the system has no periodic solutions by Theorem 4.1.

Now let us consider another $(3.1)_1$ with $a_{ii} = -1$, $a_{12} = -1$, $a_{21} = -2$, $\gamma = -2$, $e_1 = 4$, $e_2 = 3$ $(i = 1,2)$. Since they satisfy (a.2) in Theorem 3.1, the solution is bounded for any $\alpha > 0$. Since (4.5) is not satisfied, Theorem 4.1 cannot exclude the possibility of the existence of periodic solutions for $(3.1)_1$. In fact, the following shows that $(3.1)_1$ has a periodic solution for some $\alpha > 0$. The Jacobian matrix evaluated at $E_+ = (1,1,1)$ is given by

$$J_+ = \begin{pmatrix} -1 & -1 & -2 \\ -2 & -1 & 0 \\ \alpha & 0 & -\alpha \end{pmatrix}.$$

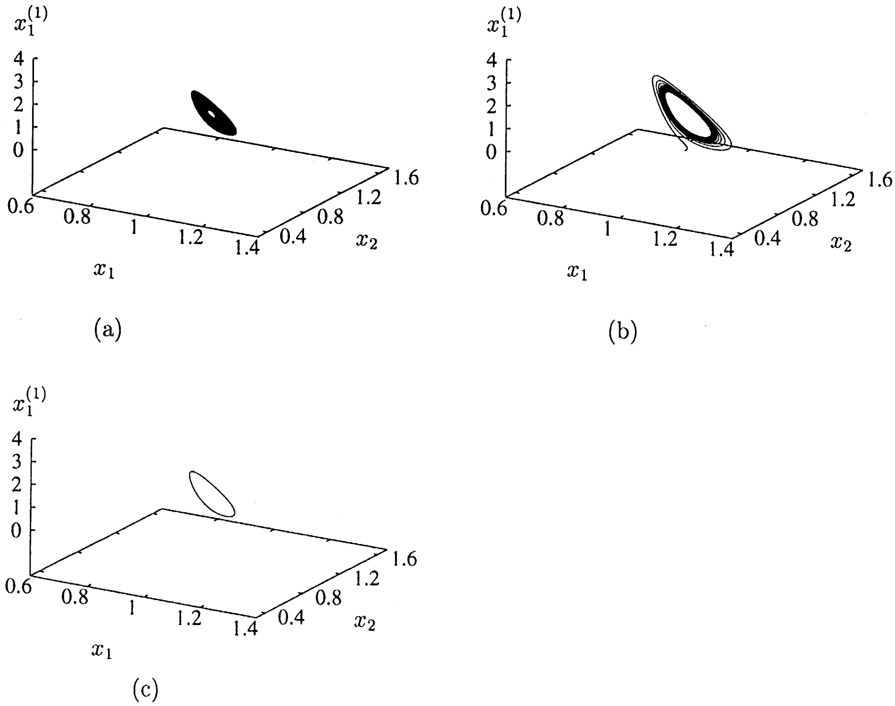

Fig. 3. A periodic solution for $(3.1)_1$ with $a_{11} = a_{22} = -1$, $a_{12} = -1$, $a_{21} = -2$, $\gamma = -2$, $e_1 = 4$, $e_2 = 3$ and $\alpha = 0.275$. (a) the trajectory with the initial value $(0.989, 1.04, 1)$, very close to $E_+ = (1, 1, 1)$, for $t \in [0, 100000]$; (b) the trajectory with the initial value $(1.2, 0.5, 1.2)$ for $t \in [0, 100000]$; (c) the trajectory with the same initial value as (b) but for $t \in [100000, 1100000]$.

Since $a_0 = 2 + \alpha > 0$, $a_1 = 4\alpha - 1$, $a_2 = \alpha$ (see Section 3) and $a_0 a_1 - a_2 = 4\alpha^2 + 6\alpha - 2$, for system $(3.1)_1$, E_+ is locally asymptotically stable for $\alpha > (\sqrt{17} - 3)/4$ and unstable for $\alpha < (\sqrt{17} - 3)/4$. Choose $\alpha = 0.275$. Fig. 3 shows that $(3.1)_1$ has a periodic solution by Hopf bifurcation.

Appendix

The definition of the second additive compound matrix can be found in [5]. Let $A = (a_{ij})$ be an $m \times m$ matrix. For $m = 2, 3, 4$ its second additive compound matrix is

$m = 2$:

$$a_{11} + a_{22} = \mathrm{tr}(A).$$

$m = 3$:

$$\begin{pmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{pmatrix}.$$

$m = 4$:

$$\begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix}.$$

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