Wave trains bifurcating from Poiseuille flow in viscous compressible fluid

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1. Introduction

- $\rho = \rho(x,t)$, $v = (v^1(x,t), \cdots, v^n(x,t))$, $t \geq 0$, $x \in \mathbf{R}^n$ $(n \geq 2)$.

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = \rho \boldsymbol{g}. \end{cases}$$

- $P = P(\rho)$: pressure; smooth in ρ,

$$P'(\rho_*) > 0 \ \text{for a constant} \ \rho_* > 0$$

- μ, μ': const's,

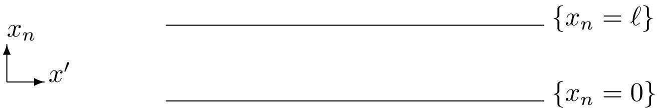
$$\mu > 0, \ \frac{2}{n}\mu + \mu' \geq 0$$

- $\boldsymbol{g}$: external force

- Quasilinear hyperbolic-parabolic system

Infinite Layer

- $\Omega_\ell = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ 0 < x_n < \ell\}$

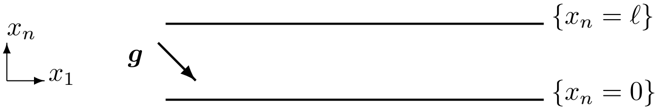


- B.C.:

$$v|_{x_n=0,\ell} = 0.$$

Parallel Flow

- $\boldsymbol{g} = (g^1(x_n), 0, \cdots, 0, g^n(x_n))$; bounded smooth
- (B.C) $v|_{x_n=0,\ell} = 0$



$\Rightarrow \exists$ smooth stationary flow $u_s(x_n) = (\rho_s(x_n), v_s(x_n))$:

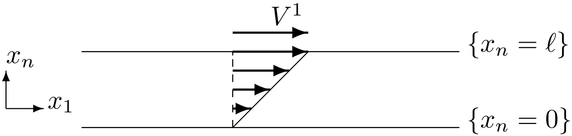
$$\inf_{x_n \in [0,\ell]} \rho_s(x_n) > 0, \quad \frac{1}{\ell}\int_0^\ell \rho_s(x_n)\,dx_n = \rho_*,$$

$$v_s = (v_s^1(x_n), 0, \cdots, 0)$$

E.g.,

plane Couette flow

- $\boldsymbol{g} = \mathbf{0}$,
- $v|_{x_n=\ell} = (V^1, 0, \cdots, 0)$, $v|_{x_n=0} = 0$

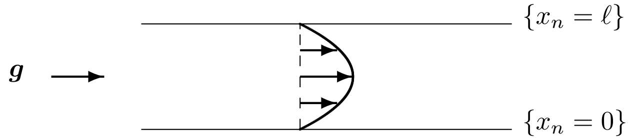


$$\rho_s = \rho_* > 0,$$

$$v_s = \left(\frac{V^1}{\ell} x_n, 0, \cdots, 0 \right).$$

Poiseuille flow

- $\boldsymbol{g} = (g^1, 0, \cdots, 0)$ ($g^1 \equiv \text{const.} \neq 0$)



$$\overline{\rho}_s = \rho_*,$$

$$\overline{v}_s = \left(\frac{\rho_* g^1}{2\mu} x_n(\ell - x_n), 0, \cdots, 0 \right).$$

3. Stability of Poiseuille flow under incompressible perturbations

- Incompressible Navier-Stokes equation:

$$\begin{cases} \operatorname{div} v = 0, \\ \rho_*(\partial_t v + v \cdot \nabla v) - \mu \Delta v + \nabla p = \rho_* \boldsymbol{g}. \end{cases}$$

- $w = v - v_s$: perturbation

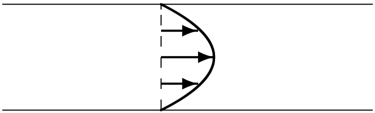
$$\begin{cases} \operatorname{div} w = 0, \\ \partial_t w - \nu \Delta w + v_s \cdot \nabla w + w \cdot \nabla v_s + w \cdot \nabla w + \nabla p = 0. \end{cases}$$

- Unconditional stability: Reynolds number $\mathrm{R} = \frac{1}{\nu}$ $(\mathrm{R} = \frac{\rho_* \ell V}{\mu})$

$$\mathrm{R} \leq \exists\, \mathrm{R}_0, \quad \Rightarrow \quad |w(t)|_{L^2} \leq e^{-\delta_0 t} |w_0|_{L^2}.$$

(Energy method + Poincaré inequality)

Poiseuille flow



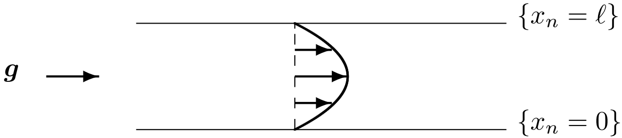
$$\rho_s = \rho_*,$$
$$v_s = \left(\frac{\rho_* g^1}{2\mu} x_n(\ell - x_n), 0, \cdots, 0 \right).$$

Incompressible case: Orszag (1971) $\exists \mathrm{R_c} \sim 5772$ s.t. $\mathrm{R} = \frac{\rho_* \ell V}{\mu}$,

$$\mathrm{R} < \mathrm{R_c} \ \Rightarrow \ \text{exponentially stable under small perturbations}$$
$$\mathrm{R} > \mathrm{R_c} \ \Rightarrow \ \text{unstable}$$

4. Stability of Poiseuille flow under compressible perturbations

- $\boldsymbol{g} = (g^1, 0, \cdots, 0)$ $(g^1 \equiv \text{const.} \neq 0)$



$$\overline{\rho}_s = \rho_*,$$

$$\overline{v}_s = \left(\frac{\rho_* g^1}{2\mu} x_n(\ell - x_n), 0, \cdots, 0 \right).$$

. Equations for Perturbation

- Non-dimensionalization

$$x = \ell \tilde{x},\ t = \frac{\ell}{V_0}\tilde{t},\ v = V_0 \tilde{v},\ \rho = \rho_* \tilde{\rho},\ P = \rho_* V_0^2 \tilde{P}$$

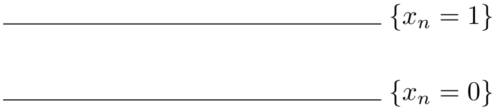
$$V_0 = \frac{\rho_* g^1 \ell^2}{\mu}$$

- $\rho_* \mapsto 1$, $\overline{v}_s \mapsto v_s = (v_s^1, 0, \cdots, 0)$, $v_s^1 = \frac{1}{2} x_n (1 - x_n)$
- $u(t) = (\phi(t), w(t)) = (\gamma^2(\rho(t) - 1), v(t) - v_s)$: perturbation

$$
(2) \quad
\begin{aligned}
&\partial_t \phi + v_s^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\, w = f^0, \\
&\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div}\, w + \nabla \phi \\
&\qquad - \tfrac{\nu}{\gamma^2} \phi \boldsymbol{e}_1 + v_s^1 \partial_{x_1} w + w^n \partial_{x_n} v_s^1 \boldsymbol{e}_1 = \boldsymbol{f},
\end{aligned}
$$

- $\boldsymbol{e}_1 = {}^\top(1, 0, \cdots, 0)$, f^0, $\boldsymbol{f}$: nonlinearities
- $\gamma = \frac{\sqrt{P'(\rho_*)}}{8V}$, $\nu = \frac{\mu}{16\rho_* \ell V}$, $\tilde{\nu} = \frac{\mu + \mu'}{16 \rho_* \ell V}$, $V = \max |\overline{v}_s| = \frac{\rho_* g^1 \ell^2}{8\mu}$.
- $\mathrm{R} = \dfrac{1}{16\nu}$: Reynolds number, $\mathrm{M} = \dfrac{1}{8\gamma}$: Mach number
- $\Omega_\ell \to \Omega \equiv \Omega_1$

$$\Omega = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ 0 < x_n < 1\}$$



- Boundary condition:

$$w|_{x_n=0,1} = 0,$$

- Initial condition:

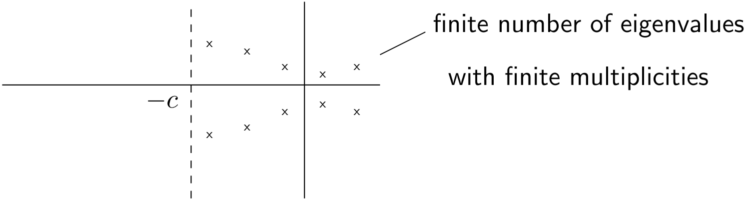
$$(\phi, w)|_{t=0} = (\phi_0, w_0).$$

4-1. Stability under spatially periodic perturbations

Iooss–Padula (1998): Linear stability of parallel flow in a cylindrical domain under spatially periodic perturbations

- $$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0.$$
- Structure of the spectrum of the linearized operator :

$$\sigma(-L) \cap \{\lambda; \mathrm{Re}\,\lambda > -c\} = \{\text{finite number of eigenvalues}\}$$



4-2. Stability under local perturbations (non-periodic, decaying at infinity)

Theorem 1 (Y.K., 2012)

$\nu \geq \exists \nu_0 > 0$, $\gamma \geq \exists \gamma_0 > 0$ (i.e., $\mathrm{R} \leq \exists \mathrm{R}_0$, $\mathrm{M} \leq \exists \mathrm{M}_0$),

$\|u_0\|_{(H^m \cap L^1)(\Omega)} \ll 1 \ (m \geq [n/2]+1)$,

$\Rightarrow$

$$\|u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}}) \quad (t \to \infty)$$

$$\|u(t) - \sigma(t) u^{(0)}\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} \eta_n(t)) \quad (t \to \infty),$$

where $u^{(0)} = u^{(0)}(x_n)$; $\sigma = \sigma(x', t)$:

$$n \geq 3: \quad \partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_1 \partial_{x_1} \sigma = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n$$

$$\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2, \ \eta_n(t) = 1 \ (n \geq 4), \ \eta_3(t) = \log(1+t)$$

$$n = 2: \quad \partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + a_1 \partial_{x_1} \sigma + a_2 \partial_{x_1}(\sigma^2) = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2,$$

$$\eta_2(t) = t^{\delta} \ (\forall \delta > 0).$$

5. Instability of Poiseuille flow (joint work with T. Nishida)

- 2D spatial-periodic perturbations ($\frac{2\pi}{\alpha}$-periodic in x_1)

$$\Omega_\alpha = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right) \times (0,1)$$

Linearized Problem

$$\partial_t u + Lu = 0, \quad u = \begin{pmatrix} \phi \\ w \end{pmatrix}.$$

- Linearized operator L on $L^2(\Omega_\alpha)$

$$L = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \mathrm{div} \\ \nabla & -\nu\Delta - \tilde{\nu}\nabla \mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2}\boldsymbol{e}_1 & v_s^1\partial_{x_1} + (\partial_{x_2} v_s^1)\boldsymbol{e}_1{}^\top\boldsymbol{e}_2 \end{pmatrix},$$

$$D(L) = \left\{ u = {}^\top(\phi, w) \in L^2(\Omega_\alpha) : \, w \in H^1_{0,per}(\Omega_\alpha), \, Lu \in L^2(\Omega_\alpha) \right\}.$$

Here $\boldsymbol{e}_1 = {}^\top(1,0)$, $\boldsymbol{e}_2 = {}^\top(0,1) \in \mathbf{R}^2$.

Theorem 2 (T. Nishida - Y.K., 2014)

There are constants $r_0 > 0$, $\eta_0 > 0$ such that if $\alpha \le r_0$, then

$$\sigma(-L) \cap \left\{\lambda \in \mathbf{C} : |\lambda| \le \eta_0\right\} = \{\lambda_{\alpha,m} : |m| = 0, 1, \cdots, m_1\}$$

for some $m_1 \in \mathbf{N}$. Here $\lambda_{\alpha,m}$ are simple eigenvalues of $-L$;
as $|\alpha m| \to 0$,

$$\lambda_{\alpha,m} = -\frac{i}{6}\alpha m + \kappa_0 \alpha^2 m^2 + O(|\alpha m|^3),$$

$$\kappa_0 = \frac{1}{12\nu}\left[\left(\frac{1}{280} - \gamma^2\right) - \frac{\nu}{30\gamma^2}(3\nu + \nu')\right].$$

Therefore, if

$$\gamma^2 < \frac{1}{280} \quad \text{and} \quad 30\gamma^2\left(\frac{1}{280} - \gamma^2\right) > \nu(3\nu + \nu'),$$

then $\kappa_0 > 0$ and Poiseuille flow is unstable.

Remark 1

Instability condition: Reynolds number $\mathrm{R} = \frac{1}{16\nu}$, Mach number $\mathrm{M} = \frac{1}{8\gamma}$

$$\mathrm{M} > \sqrt{\frac{35}{8}} \sim 2.09, \quad \frac{1}{35} - \frac{1}{8M^2} > \frac{M^2}{15R}\left(\frac{3}{R} + \frac{1}{R'}\right).$$

For example, if

$$\mathrm{M} = 2.5, \quad \mathrm{R} = \frac{173}{16} \sim 10.81, \quad \frac{1}{\mathrm{R}'} = -\frac{2}{3\mathrm{R}} \quad (\text{i.e., } \nu' = -\tfrac{2\nu}{3}),$$

then instability condition is satisfied, and plane Poiseuille flow is unstable.

Remark 2

- Incompressible case:

Orszag (1971) : Critical Reynolds number $\mathrm{R_c} \sim 5772$

$$\mathrm{R} < \mathrm{R}_c \;\Rightarrow\; \text{stable under small perturbations}$$
$$\mathrm{R} > \mathrm{R}_c \;\Rightarrow\; \text{unstable}$$

6. Bifurcating traveling waves (Wave Trains) (joint work with T.Nishida)

2D flow: $x = (x_1, x_2)$, $\frac{2\pi}{\alpha}$-periodic in x_1, $0 < x_2 < 1$

- $m = \pm 1$

$$\lambda_{\alpha,\pm 1} = \mp \frac{i}{6}\alpha + \kappa_0 \alpha^2 + O(|\alpha|^3),$$

$$\kappa_0 = \frac{1}{12\nu}\left[\left(\frac{1}{280} - \gamma^2\right) - \frac{\nu}{30\gamma^2}(3\nu + \nu')\right].$$

- γ: fix $\frac{1}{280} - \gamma^2 > 0$
- $\exists \nu_1 > 0$ such that $\kappa_0 < 0$ for $\nu = \nu_1$ and $\mathrm{Re}\,\lambda_{\alpha,\pm 1} < 0$.
- ν decreases from ν_1, then $\lambda_{\alpha,\pm 1}$ cross the imaginary axis at some $\nu = \nu_0$.
- ν: Bifurcation parameter

Notation:

$$\lambda_{\alpha,\pm 1} = \lambda_{\alpha,\pm 1}(\nu), \qquad L = L_\nu.$$

- For each $0 < \alpha \ll 1$, there exists $\nu_0 > 0$ such that

$$\begin{aligned} \nu > \nu_0 \ &\Leftrightarrow\ \mathrm{Re}\,\lambda_{\alpha,\pm 1}(\nu) < 0; \\ \nu = \nu_0 \ &\Leftrightarrow\ \mathrm{Re}\,\lambda_{\alpha,\pm 1}(\nu_0) = 0; \\ \nu < \nu_0 \ &\Leftrightarrow\ \mathrm{Re}\,\lambda_{\alpha,\pm 1}(\nu) > 0. \end{aligned}$$

Assumption:

$$\sigma(-L_{\nu_0}) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{\lambda_{\alpha,+1}(\nu_0), \lambda_{\alpha,-1}(\nu_0)\}. \tag{1}$$

Theorem 3 (T. Nishida - Y.K., 2015)

Assume that (1) holds true. Then there is a solution branch $\{\nu, u\} = \{\nu_\varepsilon, u_\varepsilon\}$ ($|\varepsilon| \ll 1$) such that

$$\begin{aligned}
&\nu_\varepsilon = \nu_0 + O(\varepsilon),\\
&u_\varepsilon = u_\varepsilon(x_1 - c_\varepsilon t, x_2), \quad u_\varepsilon(x_1 + \tfrac{2\pi}{\alpha}, x_2) = u_\varepsilon(x_1, x_2),\\
&u_\varepsilon(x_1, x_2) = \varepsilon \begin{pmatrix} 1 \\ \frac{1}{2\gamma^2} x_2(1 - x_2) \\ 0 \end{pmatrix} \tfrac{1}{\sqrt{2}} \cos \alpha x_1 (1 + O(\alpha)) + O(\varepsilon^2),\\
&\rho(x,t) = 1 + \tfrac{\varepsilon}{\sqrt{2}\gamma^2} \cos \alpha (x_1 - c_\varepsilon t)(1 + O(\alpha)) + O(\varepsilon^2),\\
&v(x,t) = \begin{pmatrix} \frac{1}{2} x_2(1 - x_2) \\ 0 \end{pmatrix} (1 + \tfrac{\varepsilon}{\sqrt{2}\gamma^2} \cos \alpha (x_1 - c_\varepsilon t)(1 + O(\alpha))) + O(\varepsilon^2),\\
&c_\varepsilon = \tfrac{1}{6} + O(\alpha^2) + O(\varepsilon).
\end{aligned}$$

Remark 3

Iooss and Padula showed that, for each ν*, there exists a positive number* Λ *such that the set*

$$\sigma(-L_\nu) \cap \{\lambda; \mathrm{Re}\,\lambda \geq -\Lambda\}$$

consists of a finite number of eigenvalues with finite multipilicities. Therefore, it seems very unlikely that assumption (1) *does not hold true.*