On synchronization and brain network

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1. Introduction

Theoretical investigations of weakly coupled limit cycle oscillators [6] are intensively developed over several research fields these days. For example, in network science, various network models are being taken into account.

It is noteworthy that the mathematical analysis of this field has been promoted recently. In this talk, we first introduce some of our results concerning the Kuramoto-Sakaguchi equation. Then, as an application, we show the equation of the *resting state network*, which is one of the most attractive topics in the brain network analysis these days.

2. Kuramoto-Sakaguchi equation

The Kuramoto-Sakaguchi equation is a physical model of the behavior of weakly coupled oscillators. It describes the temporal evolution of the probability density of the phase of each oscillator.

In this section, we first introduce some existing results concerning the Kuramoto-Sakaguchi equation.

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} - D\dfrac{\partial^2 \varrho}{\partial \theta^2} + \omega\dfrac{\partial \varrho}{\partial \theta} \\ \qquad + K\dfrac{\partial}{\partial \theta}\left[\varrho(\theta,\omega,t)\displaystyle\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_0^{2\pi} \sin(\phi-\theta)\varrho(\phi,\omega',t)\mathrm{d}\phi\right] = 0, \\ \qquad\qquad (\theta,\omega,t) \in \Omega\times\mathbf{R}\times(0,\infty), \\ \dfrac{\partial^i \varrho}{\partial \theta^i}\Big|_{\theta=0} = \dfrac{\partial^i \varrho}{\partial \theta^i}\Big|_{\theta=2\pi} \quad (i=0,1), \quad (\omega,t)\in\mathbf{R}\times(0,\infty), \\ \varrho\big|_{t=0} = \varrho_0(\theta,\omega), \quad (\theta,\omega)\in(0,2\pi)\times\mathbf{R}, \end{cases} \tag{1}$$

where $\Omega \equiv (0,2\pi)$, $g(\omega)$ is the probability distribution function of the natural frequency ω, D, the diffusion coefficient, and K is the coupling strength.

3. Resting state network : application of Kuramoto model

In this section, we consider another problem concerning the brain network, derived on the basis of Cabral's works [1][2] as an application of the Kuramoto theory.

In the region of the brain analysis, it is reported that a synchronous cooperation of multiple regions emerges when the individuals are at rest. They are now called as the *resting state network*. Recently, Cabral [1][2] derived a system of ordinary equation as a model of the average neuronal behavior in each region of the brain in the resting state network. It reads

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \omega_n + K\sum_{p=1}^{N} c_{np}\sin\big(\theta_p(t-\tau_{np}) - \theta_n(t)\big) \quad (n=1,2,\dots,N), \tag{2}$$

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where the unknown $\theta_n(t)$ $(n = 1, 2, \ldots, N)$ are the average phase of the neuronal firing in each region numbered n at time t; c_{np}, the coupling strength between neurons n and p; K, the global coupling strength which scales all connection strength; τ_{np}, the delay in the axon between neurons n and p, and ω_n, the natural frequency of the signal of a neuron numbered n.

In [4], we derived a Fokker-Planck equation corresponding to the equation (2), and discussed the mathematical well-posedness, stability and vanishing diffusion limit.

On the other hand, since the resting state network contains the dynamical property, the dynamics of the model has been left for further study [2].

Here, we consider a dynamical model of the resting state network. On the basis of existing models of modified versions of Kuramoto-Saguchi equation [5][7], it reads

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} + \omega \dfrac{\partial \varrho}{\partial \phi} - D\dfrac{\partial^2 \varrho}{\partial \phi^2} + \dfrac{kK(t)}{\overline{P}(t)} \dfrac{\partial}{\partial \phi}\mathcal{F}[\varrho, \varrho] = 0 \\ \qquad\qquad\qquad\qquad t > 0,\ (\phi, k, x, \omega) \in \Omega \times \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}, \\ \dfrac{\partial^i \varrho}{\partial \phi^i}\Big|_{\phi=0} = \dfrac{\partial^i \varrho}{\partial \phi^i}\Big|_{\phi=2\pi} \quad (i = 0, 1),\ t > 0, \quad (k, x, \omega) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \\ \varrho\big|_{t=0} = \varrho_0 \quad (\phi, k, x, \omega) \in \Omega \times \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}, \end{cases} \tag{3}$$

where

$$\mathcal{F}[\varrho_1, \varrho_2] \equiv \varrho_1(\phi, t; x, \omega) \int_{\mathbf{R}} G(x - y)\mathrm{d}y \int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_{\mathbf{R}_+} k' P(k')\mathrm{d}k' \int_0^{2\pi} \Gamma(\phi' - \phi)\varrho_2(\phi', \sigma(t; x, y); k', y, \omega')\, \mathrm{d}\phi',$$

$P(t;k)$ and $g(\omega)$ are the probability densities of the node degree k and natural frequency ω, respectively; D, the diffusion coefficient; $\Gamma(\cdot)$, the coupling function; $K(t)$, the coupling strength; $G(\cdot, t)$, the coupling strength between each node; $\sigma(t; x, y)$, the delay between nodes x and y, and $\overline{P}(t) \equiv \int_{\mathbf{R}_+} kP(t;k)\mathrm{d}k$ is the expected value of the degree.

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