Structure-preserving finite difference schemes for the Cahn-Hilliard equation with dynamic boundary conditions in the one-dimensional case

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Abstract. We study the following one-dimensional Cahn–Hilliard equation:

$$\text{(CH)} \qquad \begin{cases} \partial_t u - \partial_x^2 p = 0, \\ p = -\gamma \partial_x^2 u + F'(u), \qquad (x,t) \in (0,L) \times (0,T], \end{cases}$$

under two kinds of dynamic boundary condition. The unknowns $u := u(x,t)$ and $p := p(x,t)$ are the order parameter and the chemical potential, respectively, F' is some derivative of the potential F, and γ is a positive constant. The first problem, under the standard dynamic boundary condition, is of the following form:

$$\text{(DBC1)} \qquad \begin{cases} \partial_t u(0,t) + \partial_x u(0,t) = \partial_t u(L,t) - \partial_x u(L,t) = 0, \\ \partial_x p(0,t) = \partial_x p(L,t) = 0, \qquad t \in (0,T]. \end{cases}$$

The second problem, recently proposed by Goldstein–Miranville–Schimperna [4], is

$$\text{(DBC2)} \qquad \begin{cases} \partial_t u(0,t) + \partial_x p(0,t) = \partial_t u(L,t) - \partial_x p(L,t) = 0, \\ p(0,t) = \gamma \partial_x u(0,t) + F'(u(0,t)), \\ p(L,t) = -\gamma \partial_x u(L,t) + F'(u(L,t)) = 0, \qquad t \in (0,T]. \end{cases}$$

We propose structure-preserving finite difference schemes (see [2] and [3]) for problems (CH) with (DBC1) and (CH) with (DBC2), and give mathematical results such as an error estimate and unique existence of solution for the scheme, by using the energy method introduced in [5]. This result is based on a collaborated research with Prof. Takeshi Fukao (Kyoto University of Education) and Ms. Saori Wada (Ehime University), which has been submitted as [1].

Bibliography

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