On Large Exponent Behavior of Power Curvature Flow Arising in Image Processing

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Outline

- Motivation: Applications in Image Processing
- Introduction on Power Mean Curvature Flow
- Large Exponent Behavior
- Analogues and Other Applications

Motivation

Our interest is motion of a planar curve by a power of its curvature:

$$V = \kappa^\alpha = |\kappa|^{\alpha-1}\kappa,$$

where $\alpha > 0$ is given, κ is the curvature and V is the normal velocity.

[Andrews '98, '03] [Schulze '05, '06]

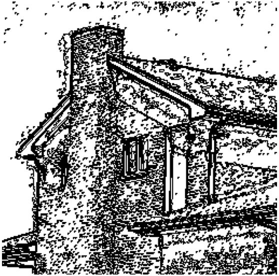
Applications in Image Processing:

Define a grey-scale image to be a function $u_0 : \mathbb{R}^2 \to \mathbb{R}$, whose range is $[0, 255]$. Let the contours move under curvature flow in the level set formulation:

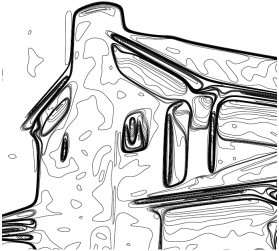
$$\begin{cases} \dfrac{u_t}{|\nabla u|} = \left[\operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)\right]^\alpha & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2. \end{cases}$$

[Alvarez-Lions-Morel '92] [Alvarez-Guichard-Lions-Morel '93] [Cao '03]

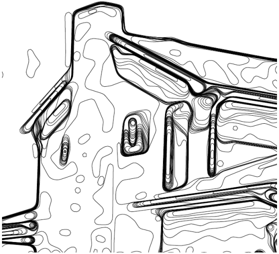
Applications in Image Processing



$\alpha = 1/3$



$\alpha = 2$



Images from F. Cao, *Geometric Curve Evolution and Image Processing*, Springer, 2003

Goal

On the choice of $\alpha > 0$ [Cao '03]

- If the purpose is shape analysis, small powers seem to be more efficient.
- If the purpose is image denoising, large powers may be more suitable.

Goal

We aim to rigorously understand the asymptotic behavior of the solution u^α when $\alpha \to 0$ and when $\alpha \to \infty$.

- See [R. M. Chen-L '16] [L, preprint] for the vanishing exponent case ($\alpha \to 0$).
- We discuss the large exponent case ($\alpha \to \infty$) in this talk.

Power Curvature Flow

We focus on the case $\underline{n=2}$ for simplicity. Consider

$$(\mathrm{PCF}_\alpha) \quad \begin{cases} u_t - |\nabla u| \left[\operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)\right]^\alpha = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^2 . \end{cases}$$

Existence and uniqueness of viscosity solutions u^α are due to

- [Chen-Giga-Goto '91] [Evans-Spruck '91] for $\alpha = 1$;
- [Ishii-Souganidis '95] for a general $\alpha > 0$.

Restriction on the class of test functions [Ishii-Souganidis '95]

A function $\varphi \in C^2(\mathbb{R}^2 \times (0,\infty))$ is called **admissible** if

$$|\varphi(x,t) - \varphi(x_0,t_0) - \varphi_t(x_0,t_0)(t-t_0)| \le f(|x-x_0|) + o(|t-t_0|),$$

holds near (x_0,t_0) with $\nabla\varphi(x_0,t_0)=0$, where $f \in C^2([0,\infty))$ satisfies

$$f(0) = f'(0) = 0, \quad f''(r) > 0 \text{ for } r > 0, \quad \lim_{r\to 0} \frac{f'(r)}{r^\alpha} = 0. \quad \left(f(r) = |r|^{\alpha+2}\right)$$

Well-posedness and Additional Properties

$$u_t - |\nabla u| \left[\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right]^\alpha = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty).$$

Existence and uniqueness [Ishii-Souganidis '95]

If u_0 is Lipschitz in $\mathbb{R}^2$, then for every $\alpha > 0$ there exists a unique viscosity solution u^α of (PCF$_\alpha$). Moreover, the **comparison principle** holds.

Lipschitz preserving

If u_0 is **Lipschitz**, then $u^\alpha(\cdot, t)$ is **Lipschitz uniformly** for all $\alpha > 0$ and $t \geq 0$.

Convexity preserving

If u_0 is **quasiconvex** in $\mathbb{R}^2$ ($\{x : u_0(x) \leq c\}$ is convex for any $c \in \mathbb{R}$), then $u^\alpha(\cdot, t)$ is also **quasiconvex** for any $\alpha > 0$ and $t \geq 0$.

$$\text{Quasiconvexity} \quad \Rightarrow \quad \operatorname{div} \left(\frac{\nabla u^\alpha}{|\nabla u^\alpha|} \right) \geq 0 \quad \Rightarrow \quad (u^\alpha)_t \geq 0 \quad \Rightarrow \quad u^\alpha \geq u_0$$

Heuristics as $\alpha \to \infty$

Let $\alpha \to \infty$ in (PCF_α). Formally, we have

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lim_{\alpha\to\infty}\left(\frac{u_t}{|\nabla u|}\right)^{\frac{1}{\alpha}} = \operatorname{sgn}\left(\frac{u_t}{|\nabla u|}\right) \in [-1, 1] \quad \text{in } \mathbb{R}^2 \times (0, \infty).$$

Example (A radially symmetric case)

If $u_0(x) = h(|x|)$ with $h : [0, \infty) \to \mathbb{R}$ Lipschitz and nondecreasing, then

$$u^\alpha(x, t) = h\left(\left(|x|^{\alpha+1} + (\alpha + 1)t\right)^{\frac{1}{\alpha+1}}\right)$$

is the unique solution of (PCF_α). Hence, for any $(x, t) \in \mathbb{R}^2 \times (0, \infty)$,

$$\lim_{\alpha\to\infty} u^\alpha(x, t) = \begin{cases} h(1) & \text{if } |x| \le 1 \\ h(|x|) & \text{if } |x| > 1 \end{cases} = \max\{u_0(x), h(1)\}.$$

An Obstacle Problem

Assume $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is **Lipschitz**, **quasiconvex** and **coercive**. We need to study

$$\text{(OP)} \qquad \min\left\{-\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right)+1,\ U-u_0\right\}=0 \quad \text{in } \mathbb{R}^2.$$

Difficulties

This obstacle problem is different from classical ones because of

- its very **strong singularity** at $\nabla U = 0$, where we cannot test;
- the **unbounded** domain and **unbounded** obstacle.

Geometric interpretation of a solution U

- $U \geq u_0$ in $\mathbb{R}^2$;
- The curvature of level curves of U is bounded from above by 1.
- The curvature of level curves of U is precisely 1 wherever $U > u_0$.

Definition of Solutions to the Obstacle Problem

Consider

$$\text{(OP)} \qquad \min\left\{-\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right)+1,\ U-u_0\right\}=0 \quad \text{in } \mathbb{R}^2,$$

Definition of viscosity solutions

An **upper semicontinuous** $U:\mathbb{R}^2\to\mathbb{R}$ is a **subsolution** of (OP) if

- $U\geq u_0$ in $\mathbb{R}^2$;
- for any $\varphi\in C^2(\mathbb{R}^2)$ and $x_0\in\mathbb{R}^2$ such that $U-\varphi$ attains a **maximum** at x_0,

$$\min\left\{-\operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)+1,\ U-u_0\right\}\leq 0 \quad \text{at } x_0 \quad \text{provided } \nabla\varphi(x_0)\neq 0.$$

Existence theorem [L–Yamada, preprint]

Assume that u_0 is Lipschitz. Then V is a solution of (OP), where

$$V(x)=\inf\{U(x) : U \text{ is a supersolution of (OP)}\}, \quad x\in\mathbb{R}^2.$$

Definition of Solutions to the Obstacle Problem

Consider

$$(\mathrm{OP}) \qquad \min\left\{-\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right)+1,\ U-u_0\right\}=0 \quad \text{in } \mathbb{R}^2,$$

Definition of viscosity solutions

A **lower semicontinuous** $U:\mathbb{R}^2\to\mathbb{R}$ is a **supersolution** of (OP) if

- $U\geq u_0$ in $\mathbb{R}^2$;
- for any $\varphi\in C^2(\mathbb{R}^2)$ and $x_0\in\mathbb{R}^2$ such that $U-\varphi$ attains a **minimum** at x_0,

$$\min\left\{-\operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)+1,\ U-u_0\right\}\geq 0 \quad \text{at } x_0 \quad \text{provided } \nabla\varphi(x_0)\neq 0.$$

Existence theorem [L–Yamada, preprint]

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$$V(x)=\inf\{U(x): U \text{ is a supersolution of (OP)}\}, \quad x\in\mathbb{R}^2.$$

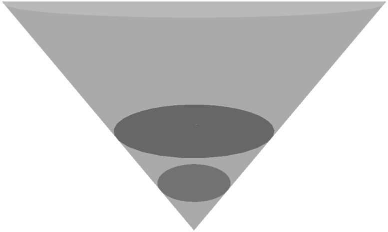
Nonuniqueness of Solutions

Example (Nonuniqueness)

Let the initial value $u_0(x) = |x|$ for any $x \in \mathbb{R}^2$. Then for any $c \geq 1$,

$U_c = \max\{u_0, c\}$ is a solution of

$$\text{(OP)} \qquad \min\left\{-\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right) + 1,\ U - u_0\right\} = 0 \quad \text{in } \mathbb{R}^2.$$



Tests are lost for U_c at the level c.

Our example shows that for any $t > 0$

$\lim_{\alpha\to\infty} u^\alpha(\cdot, t) = V = U_1 \quad (c = 1)$.

Recall

$V(x) = \inf\{U(x) : U \text{ is a supersol. of (OP)}\}$.

Large Exponent Limit

Theorem [L–Yamada, preprint]

Assume that u_0 is **Lipschitz**, **coercive** and **quasiconvex** in $\mathbb{R}^2$. Let u^α be the unique viscosity solution of (PCF$_\alpha$). Then $u^\alpha \to V$ locally uniformly in $\mathbb{R}^2 \times (0, \infty)$ as $\alpha \to \infty$, where for any $x \in \mathbb{R}^2$,

$$V(x) = \inf\{U(x) : U \text{ is a supersolution of (OP)}\}.$$

- By **Lipschitz** preserving, we may take a subsequential limit V satisfying
$$\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right) \leq 1.$$
- By **quasiconvexity** preserving, we have $u^\alpha \geq u_0$, which implies $V \geq u_0$ in $\mathbb{R}^2$.
- For any supersolution U of (OP) and $\sigma \in (0, 1)$, by **coercivity** of u_0,
$$W^\alpha(x, t) = U(\sigma x) + C_1(1 - \sigma) + C_2\sigma^\alpha t$$
is a supersolution of (PCF$_\alpha$), which by comparison implies $u^\alpha \leq W^\alpha$.
- Letting $\alpha \to \infty$ and then $\sigma \to 1$, we have $V \leq U$ in $\mathbb{R}^2$.

Selection via a Variational Approach

The large exponent limit selects the minimal solution of

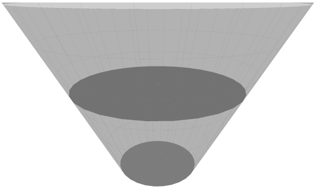
$$\text{(OP)} \qquad \min\left\{-\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right)+1,\ U-u_0\right\}=0 \quad \text{in } \mathbb{R}^2.$$

A variational interpretation

Assume $u_0 \geq 0$. Fix $R \gg 1$ and minimize

$$J_R[U]=\int_{B_R} |\nabla U(x)|+U(x)\,dx=\|U\|_{W^{1,1}(B_R)}$$

among all Lipschitz U with $U \geq u_0$ in $\mathbb{R}^2$ and $U=u_0$ in $\mathbb{R}^2 \setminus B_R$.



Example: Comparing U_c $(1 \leq c < R)$, we see

$$J_R[U_c]=\frac{\pi}{3}\left(3R^2+2R^3\right)+\frac{\pi}{3}\left(c^3-3c^2\right).$$

$$\min_{c\geq 1} J_R[U_c]=J_R[U_2] < J_R[U_1]=J_R[V].$$

Variational solution $\neq$ Viscosity (Perron) solution

An Analogue for the Power Heat Equation

- Large exponent behavior can also be considered for other evolution problems such as

$$\text{(PH)} \quad \begin{cases} u_t - |u_{xx}|^{\alpha-1} u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

- It is the space integral of parabolic p-Laplace equation ($p = \alpha + 1$):

  Suppose that the unique solution of (PH) is u^α. Let $w^\alpha = u^\alpha_x$.

  Then w^α solves

$$w_t - (|w_x|^{\alpha-1} w_x)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

- Large exponent behavior give an alternative approach to asymptotics of 1D parabolic p-Laplacian as $p \to \infty$.

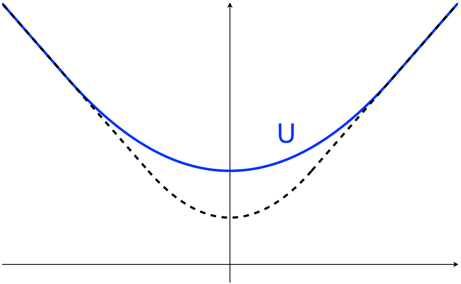
Slow/fast Diffusions and Unbounded Collapsing Sandpiles

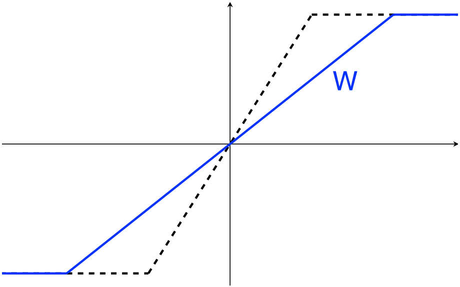
As $\alpha \to \infty$, the limit of $u^\alpha(\cdot, t)$ $(t > 0)$ is the minimal supersolution U of

$$\min\{-U_{xx} + 1, U - u_0\} = 0 \quad \text{in } \mathbb{R}$$

provided that u_0 is Lipschitz, coercive and convex in $\mathbb{R}$.

$W = U_x$ solves the limit of parabolic p-Laplace equation as $p \to \infty$.
It models collapse of unstable sandpiles [Evans-Feldman-Gariepy '97].





Summary

Conclusions:

- The large exponent limit is the minimal supersolution to a nonlinear obstacle problem in the whole space for a convex initial value (obstacle). Uniqueness of solutions to the obstacle problem fails in general due to the loss of tests at the vanishing gradient.

- Similar results hold for the heat operator of power type and find applications in the study of collapsing sandpiles.

Future Problem:

- How can we find the limit when the initial value is not convex?

Thank you for your kind attention!