Direct and inverse bifurcation problems for semilinear equations

Tetsutaro Shibata
Laboratory of Mathematics, Graduate School of Engineering
Hiroshima University

November 17, 2017

「数学と現象：Mathematics and Phenomena in Miyazaki 2017」

宮崎大学（木花キャンパス）工学部

Outline

Introduction

We consider the following nonlinear eigenvalue problems

$$-u''(t) = \lambda\left(u(t) + g(u(t))\right), \quad t \in I =: (-1, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1) = u(1) = 0, \tag{1.3}$$

where $g(u) \in C(\bar{\mathbb{R}}_+)$ and $\lambda > 0$ is a parameter.

It is well known (cf. [T. Laetsch, 1970]) that if

$$u + g(u) > 0 \qquad \text{for} \quad u > 0,$$

then by **time-map method**, we find that λ is parameterized by using $\alpha = \|u\|_\infty$, such as $\lambda = \lambda(\alpha)$ and is a continuous function of $\alpha > 0$. Since λ depends on g, we write

$$\lambda = \lambda(g, \alpha).$$

One of the nonlinear terms $g(u)$ we are interested in is

$$g_1(u) = \sin \sqrt{u}.$$

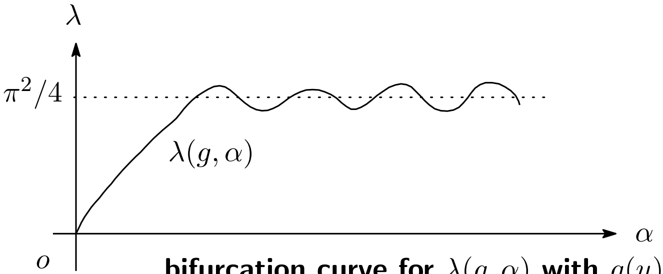
In this case, the equation (1.1)–(1.3) has been proposed in Cheng (2002) as a model problem which has arbitrary many solutions near $\lambda = \pi^2/4$.

Theorem 1.0.([Cheng, 2002]) *Let $g(u) = \sin \sqrt{u}$ ($u \geq 0$). Then for any integer $r \geq 1$, there is $\delta > 0$ such that if $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, then (1.1)–(1.3) has at least r distinct solutions.*

・Certainly, Theorem 1.0 gives us the imformation about the solution set of (1.1)–(1.3), and we expect that $\lambda(\alpha)$ oscillates and intersects the line $\lambda = \pi^2/4$ infinitely many times as $\alpha \to \infty$.

・So we expect that the bifurcation curve for g_1 is as follows.

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$



bifurcation curve for $\lambda(g, \alpha)$ with $g(u) = \sin\sqrt{u}$

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$

- The first purpose here is to prove the expectation above is valid.

- Precisely, we establish the asymptotic formula for $\lambda(g, \alpha)$ as $\alpha \to \infty$, which gives us the well understanding why $\lambda(g, \alpha)$ intersect the line $\lambda = \pi^2/4$ infinitely many times.

- We also obtain the asymptotic formula for $\lambda(g, \alpha)$ as $\alpha \to 0$. These two formulas clarify the total structure of $\lambda(g, \alpha)$.

Asymptotic length of bifurcation curve

We also consider the **asymptotic length** of $\lambda(g, \alpha)$ $(\alpha \gg 1)$ defined by

$$L(g, \alpha) := \int_{\alpha}^{2\alpha} \sqrt{1 + (\lambda'(g, s))^2} ds. \tag{1.4}$$

In particular, we are interested in $g(u)$, which satisfies

$$L(g, \alpha) = \alpha + o(\alpha), \quad (\alpha \to \infty). \tag{1.5}$$

This notion will be used to **propose a new concept** of inverse bifurcation problem.

Global behavior of bifurcation curve for $g(u) = \sin\sqrt{u}$

Theorem 1.1 ([17]). *Let* $g(u) = g_1(u) = \sin\sqrt{u}$*. Then as* $\alpha \to \infty$*,*

$$\lambda(g_1, \alpha) = \frac{\pi^2}{4} - \pi^{3/2}\alpha^{-5/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-5/4}), \quad (1.6)$$

$$\lambda'(g_1, \alpha) = \frac{1}{2}\pi^{3/2}\alpha^{-7/4}\sin\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-7/4}), \quad (1.7)$$

$$L(g_1, \alpha) = \alpha + \frac{1}{40}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + o(\alpha^{-5/2}). \quad (1.8)$$

Theorem 1.2 ([17]). *Let* $g(u) = g_1(u) = \sin\sqrt{u}$.

(i) As $\alpha \to 0$, *the following asymptotic formula for* $\lambda(g_1, \alpha)$ *holds:*

$$\lambda(g_1, \alpha) = \frac{3}{4}C_1^2\sqrt{\alpha} + \frac{3}{2}C_1C_2\alpha + O(\alpha^{3/2}), \quad (1.9)$$

where

$$C_1 := \int_0^1 \frac{1}{\sqrt{1 - s^{3/2}}}ds, \quad C_2 := -\frac{3}{8}\int_0^1 \frac{1 - s^2}{\sqrt{1 - s^{3/2}}}ds. \quad (1.10)$$

Local behavior of bifurcation curve for $g(u) = \sin\sqrt{u}$

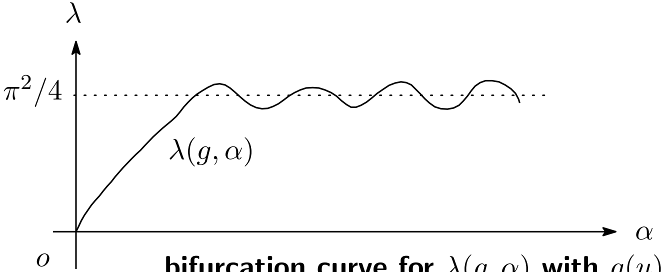
(ii) *Let* v_0 *be a unique classical solution of the following equation*

$$
\begin{aligned}
-v_0''(t) &= \frac{3}{4}C_1^2\sqrt{v_0(t)}, \quad t \in I, && (1.11)\\
v_0(t) &> 0, \quad t \in I, && (1.12)\\
v_0(-1) &= v_0(1) = 0. && (1.13)
\end{aligned}
$$

Furthermore, let $v_\alpha(t) := u_\alpha(t)/\alpha$*. Then* $v_\alpha \to v_0$ *in* $C^2(I)$ *as* $\alpha \to 0$*.*

- For the uniqueness of the positice solution of (1.12)–(1.14), we refer to A. Ambrosetti, H. Brezis, G. Cerami (1994).

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$



bifurcation curve for $\lambda(g, \alpha)$ with $g(u) = \sin\sqrt{u}$

0scillating bifurcation curve

The other nonlinear terms we treat in this talk are

$$g_2(u) = \frac{1}{2}\sin u, \tag{1.14}$$

$$g_3(u) = \sin u^2. \tag{1.15}$$

We know that the shape of $\lambda(g_2, \alpha)$ is something like Fig.2 below.

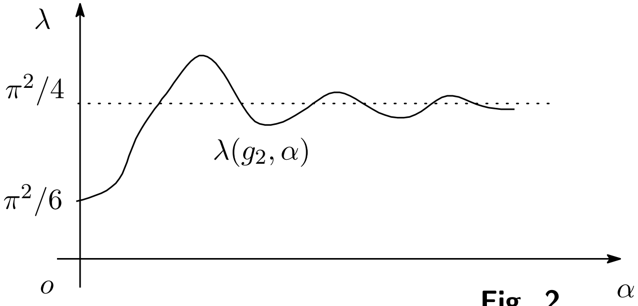


Fig. 2

Theorem 1.3 ([15]). *Let* $g(u) = g_2(u) = (1/2)\sin u$. *Then as* $\alpha \to \infty$

$$\lambda(g_2, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}), \quad (1.16)$$

$$\lambda'(g_2, \alpha) = -\frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}), \quad (1.17)$$

$$L(g_2, \alpha) = \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}). \quad (1.18)$$

Theorem 1.4 ([17]). *Let* $g(u) = g_3(u) = \sin u^2$. *Then as* $\alpha \to \infty$,

$$\lambda(g_3, \alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2}\alpha^{-2}\cos\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha^{-2}), \quad (1.19)$$

$$\lambda'(g_3, \alpha) = \frac{\pi^{3/2}}{\alpha}\sin\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha^{-1}). \quad (1.20)$$

$$L(g_3, \alpha) = \alpha + \frac{\pi^3}{8\alpha} + o(\alpha^{-1}). \quad (1.21)$$

Local behavior of the bifurcation curve for $g(u) = \sin u^2$

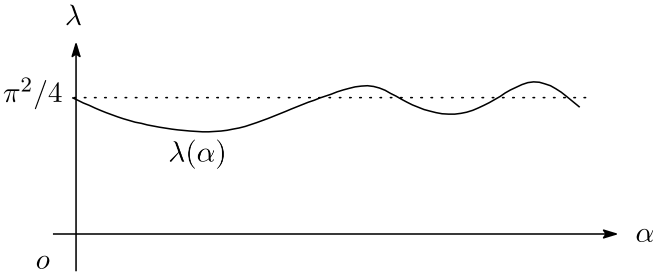
Theorem 1.5 ([17]). *Let* $g(u) = g_3(u) = \sin u^2$. *Then as* $\alpha \to 0$,

$$\lambda(g_3, \alpha) = \frac{\pi^2}{4} - \frac{1}{3}\pi A_1 \alpha + \left(\frac{1}{9}A_1^2 + \frac{1}{6}\pi A_2\right)\alpha^2 + o(\alpha^2), \quad (1.22)$$

where

$$A_1 = \int_0^1 \frac{1-s^3}{(1-s^2)^{3/2}}ds, \quad A_2 = \int_0^1 \frac{(1-s^3)^2}{(1-s^2)^{5/2}}ds. \quad (1.23)$$

Structure of the bifurcation curve for $g(u) = \sin u^2$



bifurcation curve for $\lambda(\alpha)$ **with** $g(u) = \sin u^2$

Inverse problem A

Assume that

$$g \in \Lambda := \{g \in C(\bar{\mathbb{R}}_+) : \lambda(g, \alpha) \to \pi^2/4 \text{ as } \alpha \to \infty\}$$

satisfies

$$L(g, \alpha) = \alpha + o(\alpha), \quad (\alpha \to \infty). \tag{1.24}$$

Then is it possible to distinguish g from g_i ($i = 1, 2, 3$) by the second term of $L(g, \alpha)$?

Inverse Problem A (Weak Version)

・This approach for inverse bifurcation problem seems to be **a new attempt**, and it is significant to consider whether this framework is suitable or not, since a few attempts have so far been made.

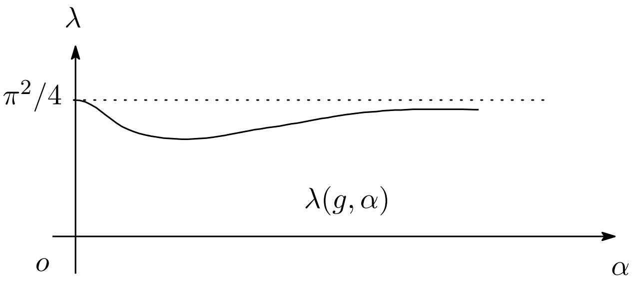
・We restrict our attention to the **'monotone' nonlinear terms** and make the simple approach to Inverse problem A.

Inverse Problem A（Weak Version)

Assume that $g(u) \in C^1(\bar{\mathbb{R}}_+)$ satisfies the following assumption (C.1).

(C.1) $g(0) = g'(0) = 0$, $g'(u) \geq 0$ for $u > 0$ and $g(u) = Cu^m$ for $u \geq 1$, where $C > 0$ and $0 < m < 1$ are constants.

Graph of $\lambda(g, \alpha)$ ($g(u)$ is "monotome" type)



bifurcation curve for $g(u) \sim Cu^m$

Answer to Inverse Problem A

Theorem 1.6 ([17]). *Let $g(u)$ satisfy (**C.1**). Then as $\alpha \to \infty$,*

$$L(g,\alpha) = \alpha + \frac{2^{2m-3}-1}{2(2m-3)} A(m)^2 \alpha^{2m-3} + o(\alpha^{2m-3}), \quad (1.25)$$

$$\lambda(g,\alpha) = \frac{\pi^2}{4} - \frac{\pi}{m+1} CC(m)\alpha^{m-1} + o(\alpha^{m-1}), \quad (1.26)$$

$$\lambda'(g,\alpha) = -\frac{m-1}{m+1}\pi CC(m)\alpha^{m-2} + o(\alpha^{m-2}), \quad (1.27)$$

where

$$A(m) := \frac{(1-m)\pi CC(m)}{1+m}, \quad C(m) = \int_0^1 \frac{1-s^{m+1}}{(1-s^2)^{3/2}} ds. \quad (1.28)$$

$$g_1(u) = \sin\sqrt{u}, \quad g_2(u) = \frac{1}{2}\sin u, \quad g_3(u) = \sin u^2,$$

and $g(u)$ is a "monotone type" $(0 < m < 1)$. Then by [15] and [17],

$$\begin{aligned} L(g_1, \alpha) &= \alpha + \frac{1}{40}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + o(\alpha^{-5/2}), \\ L(g_2, \alpha) &= \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}), \\ L(g_3, \alpha) &= \alpha + \frac{\pi^3}{8}\alpha^{-1} + o(\alpha^{-1}), \\ L(g, \alpha) &= \alpha + \frac{2^{2m-3} - 1}{2(2m-3)}A(m)^2\alpha^{2m-3} + o(\alpha^{2m-3}). \end{aligned}$$

- **We can distinguish g and g_3 by the second term of $L(g, \alpha)$.**
- **If we put $m = 1/4$ and $m = 1/2$ choose a parameter C**

appropriately, we can not distinguish g and g_1, g_2 by the second term

Proof of Theorems

$=$ **time-map**

$+$ **Asymptotic formulas for some special functions.**

・The proofs of the Theorems in this section basically depend on the time-map argument. In particular, the key tool of the proof of Theorem 1.1 is the asymptotic formula for the Bessel functions obtained by Krasikov (2016).

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. 122 (1994), 519–543.

[2] S. Cano-Casanova and J. López-Gómez, Existence, uniqueness and blow-up rate of large solutions for a canonical class of one-dimensional problems on the half-line. J. Differential Equations 244 (2008), 3180–3203.

[3] S. Cano-Casanova, J. López-Gómez, Blow-up rates of radially symmetric large solutions. J. Math. Anal. Appl. 352 (2009), 166–174.

[4] Y.J. Cheng, On an open problem of Ambrosetti, Brezis and Cerami, Differential Integral Equations 15 (2002), 1025–1044.

[5] R. Chiappinelli, On spectral asymptotics and bifurcation for elliptic operators with odd superlinear term, Nonlinear Anal. 13 (1989), 871–878.

References

[6] J. M. Fraile, J. López-Gómez and J. Sabina de Lis, On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems, J. Differential Equations 123 (1995), 180–212.
[7] A. Galstian, P. Korman and Y. Li, On the oscillations of the solution curve for a class of semilinear equations, J. Math. Anal. Appl. 321 (2006), 576–588.
[8] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products. Translated from the Russian. Translation edited and with a preface by Daniel Zwillinger and Victor Moll. Eighth edition. Elsevier/Academic Press, Amsterdam, 2015.
[9] P. Korman and Y. Li, Infinitely many solutions at a resonance, Electron. J. Differ. Equ. Conf. 05, 105–111.

References

[10] P. Korman, An oscillatory bifurcation from infinity, and from zero, NoDEA Nonlinear Differential Equations Appl. 15 (2008), 335–345.

[11] P. Korman, Global solution curves for semilinear elliptic equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2012).

[12] I. Krasikov, Approximations for the Bessel and Airy functions with an explicit error term, LMS J. Comput. Math. 17 (2014), 209–225.

[13] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20 1970/1971 1–13.

[14] T. Shibata, Asymptotic behavior of bifurcation curve for sine-Gordon type differential equation, Abstract and Applied Analysis, Volume 2012 (2012), Article ID 753857, 16 pages.

References

[15] T. Shibata, Asymptotic length of bifurcation curves related to inverse bifurcation problems, J. Math. Anal. Appl. 438 (2016), 629–642.

[16] T. Shibata, Oscillatory bifurcation for semilinear ordinary differential equations, Electron. J. Qual. Theory Differ. Equ. 2016, No. 44, 1–13.

[17] T. Shibata, Global and local structures of oscillatory bifurcation curves with application to inverse bifurcation problem, Topological Methods in Nonlinear Analysis 50 (2017), 603–622.