Time periodic problem for rotating stably stratified fluids

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The incompressible Boussinesq equations :

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = \nu \Delta v - \nabla q + \theta e_3 & t > 0,\ x \in \mathbb{R}^3, \\ \partial_t \theta + (v \cdot \nabla)\theta = \kappa \Delta \theta & t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t \geqslant 0,\ x \in \mathbb{R}^3. \end{cases}$$

- $v = (v_1(t,x),\ v_2(t,x),\ v_3(t,x))$: velocity
- $q = q(t,x)$: pressure
- $\theta = \theta(t,x)$: temperature, $\quad e_3 = (0,0,1)^T$

Boundary Conditions

- **Rotation** :

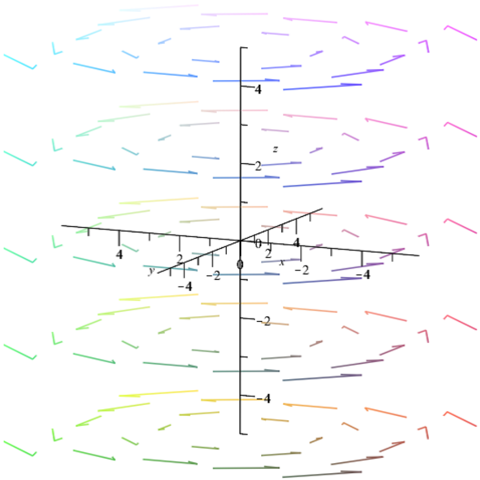
$$v \longrightarrow \Omega e_3 \times x \qquad (\,|x| \longrightarrow \infty\,) \quad (\,\Omega \in \mathbb{R}\,)$$

- **“Stable” Stratification** :

$$\theta \longrightarrow +\infty \quad (\,x_3 \longrightarrow +\infty\,), \qquad \theta \longrightarrow -\infty \quad (\,x_3 \longrightarrow -\infty\,)$$

Rotating Stably Stratified Flow (Exact stationary solution)

$$v_\Omega(x) = \Omega e_3 \times x, \quad \theta_N(x) = N^2 x_3, \quad q_{\Omega,N}(x) = \left(\Omega^2|x_h|^2 + N^2 x_3^2\right)/2$$



Rotating Stably Stratified Flow

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Perturbation

$$v(t,x) = v_\Omega(x) + \overline{v}(t,x), \quad \theta(t,x) = \theta_N(x) + \overline{\theta}(t,x), \quad q(t,x) = q_{\Omega,N}(x) + \overline{q}(t,x)$$

Rotating Coordinate

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{v}(t,x) = O(\Omega t)^T \overline{v}(t, O(\Omega t)x), \quad \tilde{\theta}(t,x) = \overline{\theta}(t, O(\Omega t)x), \quad \tilde{q}(t,x) = \overline{q}(t, O(\Omega t)x),$$

The Boussinesq equation for rotating stably stratified fluids

$$\begin{cases} \partial_t \tilde{v} + (\tilde{v}\cdot\nabla)\tilde{v} = \nu\Delta\tilde{v} - \nabla\tilde{q} + \tilde{\theta}e_3 - 2\Omega e_3 \times \tilde{v} \\ \partial_t \tilde{\theta} + (\tilde{v}\cdot\nabla)\tilde{\theta} = \kappa\Delta\tilde{\theta} - N^2 \tilde{v}_3 \\ \nabla\cdot\tilde{v} = 0 \end{cases}$$

The Boussinesq equation for rotating stably stratified fluids

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = \nu\Delta v - \nabla q + \theta e_3 - \Omega e_3 \times v + g \\ \partial_t \theta + (v\cdot\nabla)\theta = \kappa\Delta\theta - N^2 v_3 + h \\ \nabla\cdot v = 0 \\ v(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x) \end{cases}$$

- $v = (v_j(t,x))_{1\leqslant j\leqslant 3}$: velocity (around $\Omega e_3 \times x$)
- $\theta = \theta(t,x)$: temperature (around $N^2 x_3$)
- $q = q(t,x)$: pressure (around $\left(\Omega^2|x_h|^2 + N^2 x_3^2\right)/2$)
- $\Omega \in \mathbb{R}$: angular frequency, $N \geqslant 0$: buoyancy frequency
- $g = (g_j(t,x))_{1\leqslant j\leqslant 3}$, $h = h(t,x)$: Time periodic forces

AIM of this talk

- Dispersive nature of Rotation & Stable Stratification
- Application to Time Periodic Problem

Formulation

Notation:

- $u := \left(v, \dfrac{\theta}{N}\right)^T \in \mathbb{R}^4, \quad f = \left(g, \dfrac{h}{N}\right)^T \mathbb{R}^4, \quad \widetilde{\nabla} := (\nabla, 0)^T$
- $\mu := \dfrac{\Omega}{N}, \quad J_\mu := \begin{pmatrix} 0 & -\mu & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \nu\Delta v - \nabla q - \Omega e_3 \times v + \theta e_3 + g, \\ \partial_t \theta + (v \cdot \nabla)\theta = \kappa\Delta\theta - N^2 v_3 + h, \\ \nabla \cdot v = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \partial_t v + (v \cdot \nabla)v = \nu\Delta v - \nabla q - \mu N e_3 \times v + N(\theta/N)e_3 + g, \\ \partial_t(\theta/N) + (v \cdot \nabla)(\theta/N) = \kappa\Delta(\theta/N) - N v_3 + h/N, \\ \nabla \cdot v = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \partial_t u + (u \cdot \widetilde{\nabla})u = \mathrm{diag}\{\nu, \nu, \nu, \kappa\}\Delta u - \widetilde{\nabla} q - N J_\mu u + f, \\ \widetilde{\nabla} \cdot u = 0 \end{cases}$$

Formulation

$$\begin{cases} \partial_t u + (u\cdot\widetilde{\nabla})u = \mathrm{diag}\{\nu,\nu,\nu,\kappa\}\Delta u - \widetilde{\nabla} q - NJ_\mu u + f, \\ \widetilde{\nabla}\cdot u = 0 \end{cases}$$

- $\mathbb{P} := \left(\begin{array}{c|c} \left(\delta_{jk} + R_j R_k\right)_{1\leqslant j,k\leqslant 3} & 0 \\ \hline 0 & 1 \end{array}\right), \quad R_j = -\partial_j(-\Delta)^{-\frac{1}{2}}$
- $\mathbb{P} : L^2(\mathbb{R}^3)^4 \longrightarrow \left\{u \in L^2(\mathbb{R}^3)^4 \,\middle|\, \widetilde{\nabla}\cdot u = 0\right\}$
- $\mathbb{P}u = u, \quad \mathbb{P}\widetilde{\nabla} q = 0$

Applying the Helmholtz projection $\mathbb{P}$ gives

$$(\mathrm{B})_N \begin{cases} \partial_t u - \mathrm{diag}\{\nu,\nu,\nu,\kappa\}\Delta u + N\mathbb{P}J_\mu \mathbb{P}u + \mathbb{P}(u\cdot\widetilde{\nabla})u = \mathbb{P}f, \\ \widetilde{\nabla}\cdot u = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Formulation

$$(\mathrm{B})_N \begin{cases} \partial_t u - \mathrm{diag}\{\nu,\nu,\nu,\kappa\}\Delta u + N\mathbb{P}J_\mu\mathbb{P}u + \mathbb{P}(u\cdot\widetilde{\nabla})u = \mathbb{P}f, \quad \widetilde{\nabla}\cdot u = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Eigenfrequencies of $-\mathbb{P}J_\mu\mathbb{P}$

$$\sigma\left[-\widehat{\mathbb{P}J_\mu\mathbb{P}}(\xi)\right] = \left\{\pm i\frac{(\xi_1^2+\xi_2^2+\mu^2\xi_3^2)^{\frac{1}{2}}}{|\xi|}, 0, 0\right\} = \left\{\pm i\frac{|\xi_\mu|}{|\xi|}, 0, 0\right\}$$

$$\xi_\mu = (\xi_1, \xi_2, \mu\xi_3)$$

Corresponding Eigenvectors

$$a_+(\xi) = \frac{1}{\sqrt{2}|\xi_h||\xi||\xi_\mu|}\begin{pmatrix} \mu\xi_2\xi_3|\xi| + i\xi_1\xi_3|\xi_\mu| \\ -\mu\xi_1\xi_3|\xi| + i\xi_2\xi_3|\xi_\mu| \\ -i(\xi_1^2+\xi_2^2)|\xi_\mu| \\ (\xi_1^2+\xi_2^2)|\xi| \end{pmatrix}, \quad a_-(\xi) = \overline{a_+(\xi)},$$

$$a_0(\xi) = \frac{1}{|\xi_\mu|}(-\xi_2, \xi_1, 0, \mu\xi_3)^T, \quad b_0(\xi) = \frac{1}{|\xi|}(\xi, 0)^T$$

Corresponding Eigenvectors

$$a_{\pm} = \frac{1}{\sqrt{2}|\xi_h||\xi||\xi_\mu|}\begin{pmatrix} \mu\xi_2\xi_3|\xi| \pm i\xi_1\xi_3|\xi_\mu| \\ -\mu\xi_1\xi_3|\xi| \pm i\xi_2\xi_3|\xi_\mu| \\ \mp i(\xi_1^2+\xi_2^2)|\xi_\mu| \\ (\xi_1^2+\xi_2^2)|\xi| \end{pmatrix}, \quad a_0 = \frac{1}{|\xi_\mu|}\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ \mu\xi_3 \end{pmatrix}, \quad b_0 = \frac{1}{|\xi|}\begin{pmatrix} \xi \\ 0 \end{pmatrix}$$

$$P_j u_0 := \mathcal{F}^{-1}\left[\left\langle \widehat{u_0}(\xi),\, a_j(\xi)\right\rangle_{\mathbb{C}^4} a_j(\xi)\right] \quad (j = \pm,\, 0)$$

Proposition (Linear solution/semigroup on $L^2_\sigma(\mathbb{R}^3)$)

$$\nu = \kappa \implies e^{t(\nu\Delta - N\mathbb{P}J_\mu\mathbb{P})}u_0 = e^{\nu t\Delta}e^{iNtp_\mu(D)}P_+u_0 + e^{\nu t\Delta}e^{-iNtp_\mu(D)}P_-u_0 + e^{\nu t\Delta}P_0u_0$$

$$p_\mu(\xi) = \frac{|\xi_\mu|}{|\xi|}, \quad e^{\pm iNtp_\mu(D)}f(x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm iNtp_\mu(\xi)}\widehat{f}(\xi)\,d\xi$$

Remark

$$\widetilde{\nabla}\cdot u_0 = 0 \iff (\xi,\,0)^T\cdot\widehat{u_0}(\xi) = 0 \iff \langle\widehat{u_0}(\xi),\, b_0(\xi)\rangle_{\mathbb{C}^4} = 0$$

Formulation

Proposition (Linear solution/semigroup on $L^2_\sigma(\mathbb{R}^3)$)

$$\nu = \kappa \implies e^{t(\nu\Delta - N\mathbb{P}J_\mu\mathbb{P})}u_0 = e^{\nu t\Delta}e^{iNtp_\mu(D)}P_+u_0 + e^{\nu t\Delta}e^{-iNtp_\mu(D)}P_-u_0 + e^{\nu t\Delta}P_0u_0$$

Assumptions

1. $\nu = \kappa = 1$ (A1)
2. $\exists F : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^{4\times 4}$ *s.t.* $f = \widetilde{\nabla} \cdot F$ (A2)

$$\begin{cases} \partial_t u - \Delta u + N\mathbb{P}J_\mu \mathbb{P}u + \mathbb{P}(u \cdot \widetilde{\nabla})u = \mathbb{P}\widetilde{\nabla} \cdot F, \\ \widetilde{\nabla} \cdot u = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Integral Equations

$L_N := -\Delta + N\mathbb{P}J_\mu \mathbb{P}$

$$u(t) = e^{-tL_N}u_0 + \int_0^t e^{-(t-\tau)L_N}\mathbb{P}\widetilde{\nabla} \cdot (F - u \otimes u)(\tau)\, d\tau \qquad \text{(IE)}$$

Time periodic problem for NS system in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \Delta u - \nabla p + \nabla\cdot F, \\ \nabla\cdot u = 0 \end{cases}$$

- Maremonti (1991)
  - $t \in \mathbb{R}_+$, Unique existence of strong T-periodic sol.
- Kozono-Nakao (1996)
  - $t \in \mathbb{R}$
  - $u(t) = \displaystyle\int_{-\infty}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(F - u\otimes u)(\tau)\,d\tau$
  - $F \in BC(\mathbb{R}; L^{q_1}(\mathbb{R}^3)\cap L^{q_2}(\mathbb{R}^3))$: *small*, $\quad 1 < \exists q_1 < \dfrac{3}{2} < \exists q_2 < \infty$

Rem. $F_\lambda(t,x) := \lambda^2 F(\lambda^2 t, \lambda x) \implies \|F_\lambda(0)\|_{L^{\frac{3}{2}}} = \|F(0)\|_{L^{\frac{3}{2}}}$

- Yamazaki (2000)
  - $F \in BC(\mathbb{R}; L^{\frac{3}{2},\infty}(\mathbb{R}^3))$: *small*

- Geissert-Hieber-Nguyen (2016)
  - $t \in \mathbb{R}_+$
  - $u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (F - u \otimes u)(\tau)\, d\tau$
  - $F \in BC(\mathbb{R}_+; L^{\frac{3}{2},\infty}(\mathbb{R}^3))$: T-periodic, small
    $\Longrightarrow$ $\exists! u_0 \in L^{3,\infty}(\mathbb{R}^3)$,
    $\exists! u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$: T-periodic solution

Rem. $u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau, \quad f, u : T$-periodic

$$(u(t) = u(t+T)) \quad \Longrightarrow \quad (Id - e^{-TA})u_0 = \int_0^T e^{-(T-\tau)A} f(\tau)\, d\tau$$

IVP problem for rotating stably stratified fluids

- Babin-Mahalov-Nicolaenko (1999, 2001) $\mathbb{T}^3$
- Charve (2004) $\mathbb{R}^3$
- Koba-Mahalov-Yoneda (2012) $\mathbb{R}^3$

Main Result

$$\mathbb{P}F = P_+F + P_-F + P_0F$$

Theorem

- *Assume* $(A1)$, $(A2)$ *and* $\mu = \dfrac{\Omega}{N} \neq 0, \pm 1$
- $0 < s < \dfrac{1}{2}, \quad \max\left\{\dfrac{2-s}{3}, \dfrac{5+2s}{9}\right\} < \dfrac{1}{p} < \dfrac{2}{3}$

$\Longrightarrow\ \exists \delta_1 = \delta_1(\mu, s, p) > 0,\ \exists \delta_2 > 0$ *s.t.*

$\forall F \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$: T*-periodic with*

$$\begin{cases} \sup_{t>0} \|P_+F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_-F(t)\|_{L^{p,\infty}} \leqslant \delta_1 N^{\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)}, \\ \sup_{t>0} \|P_0F(t)\|_{L^{\frac{3}{2},\infty}} \leqslant \delta_2 \end{cases}$$

$\exists! u_0 \in L^{3,\infty}(\mathbb{R}^3)$ & $\exists! u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$: T*-periodic mild solution*

Dispersive Estimates for $e^{\pm itp_\mu(D)}$

Linear Propagator

$$e^{\pm itp_\mu(D)}f(x) := \mathscr{F}^{-1}\left[e^{\pm itp_\mu(\xi)}\widehat{f}(\xi)\right](x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm itp_\mu(\xi)}\widehat{f}(\xi)d\xi$$

- the phase $p_\mu(\xi) = \dfrac{|\xi_\mu|}{|\xi|}$: homogeneous of degree 0

$\Longrightarrow$ it suffices to consider the frequency localized case:

$$U_\pm(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm itp_\mu(\xi)}\psi(\xi)\widehat{f}(\xi)d\xi,$$

where $\psi \in \mathscr{S}(\mathbb{R}^3)$ satisfying

$$\operatorname{supp}\psi \subset \left\{2^{-2} \leqslant |\xi| \leqslant 2^2\right\} \quad \& \quad \psi(\xi) = 1\ (2^{-1} \leqslant |\xi| \leqslant 2)$$

Dispersive Estimates

Lemma (Dispersive Estimate)

$\forall \mu \in \mathbb{R} \setminus \{0, \pm 1\}$, $\exists C = C(\mu) > 0$ *s.t.*

$$\|U_\pm(t) f\|_{L^\infty} \leqslant C(1 + |t|)^{-\frac{1}{2}} \|f\|_{L^1}$$

for $\forall t \in \mathbb{R}$. *Also, the decay rate* $1/2$ *is sharp.*

Lemma ($L^{p,\infty}$-$L^{q,\infty}$ estimate)

$\mu \in \mathbb{R} \setminus \{0, \pm 1\}$, $s \geqslant 0$, $1 < p < q' < 2 < q < \infty$

$$\left\|(-\Delta)^{\frac{s}{2}} e^{t\Delta} e^{\pm iNtp_\mu(D)} f\right\|_{L^{q,\infty}} \leqslant C_{\mu,s,p,q}\, t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}} (1 + Nt)^{-\frac{1}{2}(1-\frac{2}{q})} \|f\|_{L^{p,\infty}}$$

Remarks

- $\mu = 0 \implies \dfrac{|\xi_\mu|}{|\xi|} = \dfrac{(\xi_1^2 + \xi_2^2 + \mu^2 \xi_3^2)^{\frac{1}{2}}}{|\xi|} = \dfrac{(\xi_1^2 + \xi_2^2)^{\frac{1}{2}}}{|\xi|}$

  :singularity along $\{(\xi_1, \xi_2) = 0\}$

- $\mu = \pm 1 \implies \dfrac{|\xi_\mu|}{|\xi|} \equiv 1$: NO dispersion

Dispersive Estimates

Theorem (Littman ('63))

- $\psi \in C_0^\infty(\mathbb{R}^d), \quad p \in C^\infty(\operatorname{supp}\psi; \mathbb{R})$
- $\operatorname{rank} \nabla^2 p(\xi) \geqslant k$ *on* $\operatorname{supp}\psi$

$\Longrightarrow \ \exists C = C(d, \psi, p) > 0$ *s.t.*

$$\left| \int_{\mathbb{R}^d} e^{ix\cdot\xi + itp(\xi)} \psi(\xi)\, d\xi \right| \leqslant C(1+|t|)^{-\frac{k}{2}}$$

The case of rotating fluids (Koh-Lee-T., '14)

- $p(\xi) = \dfrac{\xi_3}{|\xi|} \in C^\infty(\{2^{-2} \leqslant |\xi| \leqslant 2^2\}; \mathbb{R})$
- $\det \nabla^2 \left(\dfrac{\xi_3}{|\xi|} \right) = \dfrac{(\xi_1^2 + \xi_2^2)\xi_3}{|\xi|^9}, \quad \operatorname{rank} \nabla^2 \left(\dfrac{\xi_3}{|\xi|} \right) \geqslant 2$
- $\left| \int_{\mathbb{R}^3} e^{ix\cdot\xi + it\frac{\xi_3}{|\xi|}} \psi(\xi)\, d\xi \right| \leqslant C(1+|t|)^{-1} \qquad \forall (t,x) \in \mathbb{R}^{1+3}$

Dispersive Estimates

$$\nabla^2 \left(\frac{|\xi_\mu|}{|\xi|} \right) = R_\mu(\xi) + M_\mu(\xi)$$

$$R_\mu(\xi) := \frac{(\mu^2-1)\xi_3}{|\xi_\mu||\xi|^5} \begin{pmatrix} \xi_3(3\xi_1^2 - |\xi|^2) & 3\xi_1\xi_2\xi_3 & \xi_1(3\xi_3^2 - |\xi|^2) \\ 3\xi_1\xi_2\xi_3 & \xi_3(3\xi_2^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) \\ \xi_1(3\xi_3^2 - |\xi|^2) & \xi_2(3\xi_3^2 - |\xi|^2) & -3\xi_3(\xi_1^2 + \xi_2^2) \end{pmatrix}$$

$$= \frac{(\mu^2-1)\xi_3}{|\xi_\mu|} \nabla^2 \left(\frac{\xi_3}{|\xi|} \right),$$

$$M_\mu(\xi) := \frac{\mu^2-1}{|\xi|^3|\xi_\mu|^3} \begin{pmatrix} \xi_1^2\xi_3^2 & \xi_1\xi_2\xi_3^2 & -\xi_1\xi_3(\xi_1^2+\xi_2^2) \\ \xi_1\xi_2\xi_3^2 & \xi_2^2\xi_3^2 & -\xi_2\xi_3(\xi_1^2+\xi_2^2) \\ -\xi_1\xi_3(\xi_1^2+\xi_2^2) & -\xi_2\xi_3(\xi_1^2+\xi_2^2) & (\xi_1^2+\xi_2^2)^2 \end{pmatrix}$$

$$\det \nabla^2 \left(\frac{|\xi_\mu|}{|\xi|} \right) = (\mu^2-1)^3 \frac{(\xi_1^2+\xi_2^2)\xi_3^4}{|\xi|^9|\xi_\mu|^3}$$

Dispersive Estimates

$$\det \nabla^2\left(\frac{|\xi_\mu|}{|\xi|}\right) = (\mu^2-1)^3 \frac{(\xi_1^2+\xi_2^2)\xi_3^4}{|\xi|^9|\xi_\mu|^3}$$

$\chi \in C_0^\infty([0,\infty))$ s.t. $\chi(r)=1$ on $[0,1/2)$ & $\chi(r)=0$ on $[1,\infty)$
$0<c\ll 1$: small constant, $\xi_h := (\xi_1,\xi_2)$

$$U_\pm(t)f(x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm itp_\mu(\xi)}\psi(\xi)\widehat{f}(\xi)d\xi = \sum_{j=1}^{3}\mathcal{G}_\mu^j(t)f(x)$$

where

$$\mathcal{G}_\mu^1(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi\pm itp_\mu(\xi)}\left\{1-\chi\left(\frac{|\xi_h|}{c}\right)\right\}\left\{1-\chi\left(\frac{|\xi_3|}{c}\right)\right\}\psi(\xi)\widehat{f}(\xi)d\xi$$

$$\mathcal{G}_\mu^2(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi\pm itp_\mu(\xi)}\chi\left(\frac{|\xi_h|}{c}\right)\psi(\xi)\widehat{f}(\xi)d\xi$$

$$\mathcal{G}_\mu^3(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi\pm itp_\mu(\xi)}\chi\left(\frac{|\xi_3|}{c}\right)\psi(\xi)\widehat{f}(\xi)d\xi$$

Dispersive Estimates

Degenearte region I : $(\xi_1, \xi_2) = 0, \ |\xi_3| \sim 1$

$$R_\mu(0,0,\xi_3) = \frac{1-\mu^2}{|\mu|\xi_3^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_\mu(0,0,\xi_3) = 0$$

Degenearte region II : $\xi_3 = 0, \ |(\xi_1, \xi_2)| \sim 1$

$$R_\mu(\xi_1,\xi_2,0) = 0, \quad M_\mu(\xi_1,\xi_2,0) = \frac{\mu^2-1}{\xi_1^2+\xi_2^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lemma

$$\left\|\mathscr{G}_\mu^1(t) f\right\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{2}}\|f\|_{L^1}$$
$$\left\|\mathscr{G}_\mu^2(t) f\right\|_{L^\infty} \leqslant C(1+t)^{-1}\|f\|_{L^1}$$
$$\left\|\mathscr{G}_\mu^3(t) f\right\|_{L^\infty} \leqslant C(1+t)^{-\frac{1}{2}}\|f\|_{L^1}$$

Linear Estimates

Lemma (Yamazaki (2000))

$$\sup_{t>0} \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \widetilde{\nabla} \cdot F(\tau)\, d\tau \right\|_{L^{3,\infty}} \leqslant C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}}$$

By duality, the matter is reduced to the estimate

$$\int_0^\infty \|\nabla e^{t\Delta} \varphi\|_{L^{3,1}}\, dt \leqslant C \|\varphi\|_{L^{\frac{3}{2},1}}$$

Note $\int_0^\infty \|\nabla e^{t\Delta} \varphi\|_{L^3}\, dt \leqslant C \|\varphi\|_{L^{\frac{3}{2}}}$ does NOT hold :

- $\int_0^\infty \|\nabla e^{t\Delta} \varphi\|_{L^3}\, dt \simeq \|\varphi\|_{\dot{B}^{-1}_{3,1}}$
- $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{W}^{-1,3}(\mathbb{R}^3), \quad L^{\frac{3}{2}}(\mathbb{R}^3) \not\subset \dot{B}^{-1}_{3,1}(\mathbb{R}^3) \hookrightarrow \dot{W}^{-1,3}(\mathbb{R}^3)$

Lemma (Inhomogeneous estimate)

$$\mu \in \mathbb{R} \setminus \{0, \pm 1\}, \quad 0 < s < \frac{1}{2}, \quad \max\left\{\frac{2-s}{3}, \frac{5+2s}{9}\right\} < \frac{1}{p} < \frac{2}{3}$$

$$\sup_{t>0} \left\| \int_0^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau)\, d\tau \right\|_{L^{3,\infty}} \leqslant C_{\mu,s,p} N^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \sup_{t>0} \|F(t)\|_{L^{p,\infty}}$$

[Proof] $\frac{1}{q} = \frac{1}{3} + \frac{s}{3}$. By $\dot{W}^s_{q,\infty}(\mathbb{R}^3) \hookrightarrow L^{3,\infty}(\mathbb{R}^3)$ and $L^{p,\infty}$-$L^{q,\infty}$ estimate,

$$\left\| \int_0^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau)\, d\tau \right\|_{L^{3,\infty}} \leqslant C \sup_{t>0} \|F(t)\|_{L^{p,\infty}} \int_0^\infty t^{-\frac{3}{2p}} (1+Nt)^{-\frac{1}{6}+\frac{s}{3}}\, dt$$

Here, since $\frac{5+2s}{9} < \frac{1}{p} < \frac{2}{3}$,

$$\int_0^\infty t^{-\frac{3}{2p}} (1+Nt)^{-\frac{1}{6}+\frac{s}{3}}\, dt = N^{-1+\frac{3}{2p}} \int_0^\infty t^{-\frac{3}{2p}} (1+t)^{-\frac{1}{6}+\frac{s}{3}}\, dt < \infty.$$

Linear T-periodic mild solution

$$u(t) = e^{-tL_N}u_0 + \int_0^t e^{-(t-\tau)L_N}\mathbb{P}\widetilde{\nabla}\cdot F(\tau)\,d\tau \qquad (L)$$

Theorem

- *Assume* $(A1)$, $(A2)$ *and* $\mu \neq 0, \pm 1$
- $0 < s < \frac{1}{2}, \quad \max\left\{\frac{2-s}{3}, \frac{5+2s}{9}\right\} < \frac{1}{p} < \frac{2}{3}$

$\Longrightarrow$ $\forall F \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$: T*-periodic*
$\exists! u_0 \in L^{3,\infty}(\mathbb{R}^3)$ & $\exists! u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$: T*-periodic sol. to* (L)

Moreover,

$$\sup_{t>0}\|u(t)\|_{L^{3,\infty}} \leqslant CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)}\left\{\sup_{t>0}\|P_+F(t)\|_{L^{p,\infty}} + \sup_{t>0}\|P_-F(t)\|_{L^{p,\infty}}\right\} + C\sup_{t>0}\|P_0F(t)\|_{L^{p,\infty}}$$

Linear T-periodic mild solution

[Sketch of Proof]

$$\exists ! u(t) = e^{-tL_N} u_0 + \int_0^t e^{-(t-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau : T\text{-}periodic$$

$$\Longleftrightarrow \exists ! u_0 \quad s.t. \quad (Id - e^{-TL_N}) u_0 = \int_0^T e^{-(T-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau$$

Construction of u_0:

- $G := \int_0^T e^{-(T-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau$
- $a_k := \sum_{j=1}^{k-1} e^{-jTL_N} G = \int_0^{(k-1)T} e^{-(kT-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau$
- $a := w^* - \lim_{k \to \infty} a_k$ in $L^{3,\infty}(\mathbb{R}^3)$
- $u_0 := a + G$

Linear T-periodic mild solution

Construction of u_0

- $G := \int_0^T e^{-(T-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau$
- $a_k := \sum_{j=1}^{k-1} e^{-jTL_N} G = \int_0^{(k-1)T} e^{-(kT-\tau)L_N} \mathbb{P}\widetilde{\nabla} \cdot F(\tau)\, d\tau$
- $a := w^* - \lim_{k\to\infty} a_k$ in $L^{3,\infty}(\mathbb{R}^3)$, $\quad u_0 := a + G$

Formally, since $a_k = \sum_{j=1}^{k-1} e^{-jTL_N} G$,

$$(Id - e^{-TL_N})a_k = \sum_{j=1}^{k-1} \left(e^{-jTL_N} - e^{-(j+1)TL_N}\right) G = e^{-TL_N} G - e^{-kTL_N} G$$

$$\Longrightarrow \quad (Id - e^{-TL_N})a = e^{-TL_N} G \qquad (k \longrightarrow \infty)$$

$$\Longrightarrow \quad (Id - e^{-TL_N})(a + G) = G$$

Linear T-periodic mild solution

By the $L^{p,\infty}$-$L^{q,\infty}$ estimate,

$$\|e^{-jTL_N}G\|_{L^{6,\infty}} \leqslant \int_0^T \|e^{-\{(j+1)T-\tau\}L_N}\mathbb{P}\widetilde{\nabla}\cdot F(\tau)\|_{L^{6,\infty}}\,d\tau \leqslant C_T\, j^{-\frac{5}{4}} \sup_{t>0}\|F(t)\|_{L^{\frac{3}{2},\infty}}$$

Hence, $a_k = \sum_{j=1}^{k-1} e^{-jTL_N}G$ is Cauchy seq. in $L^{6,\infty}(\mathbb{R}^3)$.

$$\|a_k\|_{L^{3,\infty}} = \left\| e^{-TL_N}\int_0^{(k-1)T} e^{\{(k-1)T-\tau\}L_N}\mathbb{P}\widetilde{\nabla}\cdot F(\tau)\,d\tau \right\|_{L^{3,\infty}}$$
$$\leqslant C(1+NT)^2 \sup_{t>0}\|F(t)\|_{L^{\frac{3}{2},\infty}}$$

$$\exists\, a = s-\lim_{k\to\infty} a_k \ \ (L^{6,\infty}(\mathbb{R}^3)) = w^*-\lim_{k\to\infty} a_k \ \ (L^{3,\infty}(\mathbb{R}^3))$$

$$w^*-\lim_{k\to\infty}(Id-e^{-TL_N})a_k = (Id-e^{-TL_N})a \quad \text{in } L^{3,\infty}(\mathbb{R}^3)$$

Linear T-periodic mild solution

$$w^* - \lim_{k\to\infty} (Id - e^{-TL_N})a_k = (Id - e^{-TL_N})a \quad \text{in } L^{3,\infty}(\mathbb{R}^3)$$

Moreover,

$$\begin{aligned}&\|(Id - e^{-TL_N})a_k - e^{-TL_N}G\|_{L^{6,\infty}} \\ &\quad = \|e^{-kTL_N}G\|_{L^{6,\infty}} \leqslant C_T k^{-\frac{5}{4}} \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \longrightarrow 0 \quad (k \longrightarrow \infty).\end{aligned}$$

Hence we obtain

- $s - \lim_{k\to\infty} (Id - e^{-TL_N})a_k = e^{-TL_N}G \quad \text{in } L^{6,\infty}(\mathbb{R}^3)$
- $w^* - \lim_{k\to\infty} (Id - e^{-TL_N})a_k = e^{-TL_N}G \quad \text{in } L^{3,\infty}(\mathbb{R}^3)$

$$(Id - e^{-TL_N})(a + G) = G \quad \text{in } L^{3,\infty}(\mathbb{R}^3)$$