Long range scattering for NLS equation with critical homogeneous nonlinearity in 3d

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Introduction

Nonlinear Schrödinger equation:

(NLS) $$i\partial_t u + \Delta u = F(u), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,$$

where $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$: the unknown function.
F : homogeneous of degree $1 + \frac{2}{d}$, i.e. for any $\lambda > 0$,

$$F(\lambda u) = \lambda^{1+\frac{2}{d}} F(u).$$

Aim: Asymptotic behavior of the solution to (NLS) as $t \to \infty$.
Final state problem: For given $u_+ \in L^2(\mathbb{R}^d)$, we seek the solution to

(FS) $$\begin{cases} i\partial_t u + \Delta u = \mu|u|^{p-1}u, & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \|u(t) - u_a(t)\|_{L^2} \to 0 \quad as \quad t \to \infty. \end{cases}$$

In the most typical case, $u_a(t) = U(t)u_+$ (scattering), $U(t) = e^{it\Delta}$.

- $p > 1 + \frac{2}{d} \overset{\exists}{\Longrightarrow}$ solution of (FS) with $u_a(t) = U(t)u_+$.
- $1 < p \leq 1 + \frac{2}{d} \overset{\nexists}{\Longrightarrow}$ solution of (FS) with $u_a(t) = U(t)u_+$.
- $p = 1 + \frac{2}{d} \overset{\exists}{\Longrightarrow}$ solution of (FS) with

$$u_a(t) = U(t)\mathcal{F}^{-1} e^{-i\mu|\widehat{u}_+|^{\frac{2}{d}} \log t} \mathcal{F}u_+ \qquad \text{(modified scattering)}$$
$$= (2it)^{-d/2} e^{\frac{i|x|^2}{4t}} \widehat{u_+}\left(\frac{x}{2t}\right) e^{-i\mu\left|\widehat{u}_+\left(\frac{x}{2t}\right)\right|^{\frac{2}{d}} \log t}$$

Barab ('84), Tsutsumi-Yajima ('84), Ozawa ('91), Ginibre-Ozawa ('93)

Known results

Final state problem:

$$\begin{cases} i\partial_t u + \Delta u = F(u), & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \|u(t) - u_a(t)\|_{L^2} \to 0 & as \quad t \to \infty. \end{cases}$$

Monomial nonlinearities:

$d = 1$ $(1 + \frac{2}{d} = 3)$

$\cdot\ F(u) = u^3,\ |u|^2\overline{u},\ \overline{u}^3$ $\quad\exists$ solution of (FS) with $u_a(t) = U(t)u_+$.
Moriyama-Tonegawa-Y. Tsutsumi ('03)

$d = 2$ $(1 + \frac{2}{d} = 2)$

$\cdot\ F(u) = u^2,\ \overline{u}^2$ $\quad\exists$ solution of (FS) with $u_a(t) = U(t)u_+$.
Moriyama-Tonegawa-Y. Tsutsumi ('03)

$\cdot\ F(u) = |u|^2$ $\quad\nexists$ solution of (FS) with $u_a(t) = U(t)u_+$.
Shimomura-Y. Tsutsumi ('06)

Existence of blow-up solution even for small data.
Ikeda-Wakasugi ('13), Ikeda-Inui ('15)

Known results

Considerable examples:

$$|\mathrm{Re}\,(u)|^2\mathrm{Re}\,(u) = \left(\frac{u+\overline{u}}{2}\right)^3 = \underbrace{\frac{3}{8}|u|^2u}_{\text{resonant part}} + \underbrace{\frac{1}{8}u^3 + \frac{3}{8}|u|^2\overline{u} + \frac{1}{8}\overline{u}^3}_{\text{non-resonant part}}.$$

$$|\mathrm{Re}\,(u)|^{\frac{2}{d}}\mathrm{Re}\,(u) = \ \mathbf{?} \quad \text{when } d \neq 1.$$

Generalized Gross-Pitaevskii equation:

$$\text{(GGP)} \qquad \begin{cases} i\partial_t u + \Delta u = ||u|^2 - 1|^{p-2}(|u|^2-1)u, & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \quad u(0,x) = u_0(x), & x \in \mathbb{R}^d, \\ |u(t,x)|^2 \to 1 \quad (|x| \to \infty), \end{cases}$$

where $d \geq 1$, $p \geq 2$ ($p = 2 \Rightarrow$ Gross-Pitaevskii equation).
Putting $u = 1 + v$,

$$\text{(GGP)} \iff i\partial_t v + \Delta v = ||v|^2 + 2\mathrm{Re}\,(v)|^{p-2}(|v|^2 + 2\mathrm{Re}\,(v))(1+v).$$

For the case $p = 3$,

$$||v|^2 + 2\mathrm{Re}\,(v)|^2(|v|^2 + 2\mathrm{Re}\,(v))(1+v) \sim |\mathrm{Re}\,(v)|\mathrm{Re}\,(v).$$

cf. Masaki-Miyazaki (arxiv '16)

Idea of Masaki-Miyazaki in 1d and 2d.

We review the result by Masaki-Miyazaki (arxiv '16).

$$F(u) = |u|^{1+\frac{2}{d}} F\left(\frac{u}{|u|}\right) = |u|^{1+\frac{2}{d}} F(e^{i\theta}) = |u|^{1+\frac{2}{d}} g(\theta) \quad (e^{i\theta} = \arg u)$$

Fourier series expansion:

$$g(\theta) = \sum_{n\in\mathbb{Z}} g_n e^{in\theta}, \quad g_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta) e^{-in\theta} d\theta.$$

Therefore,

$$F(u) = |u|^{1+\frac{2}{d}} \sum_{n\in\mathbb{Z}} g_n e^{in\theta} = \sum_{n\in\mathbb{Z}} g_n |u|^{1+\frac{2}{d}-n} u^n.$$

Actually, they identified the resonant part $|\mathrm{Re}\,(u)|\mathrm{Re}\,(u)$ as

$$|\mathrm{Re}\,(u)|\mathrm{Re}\,(u) = \underbrace{\frac{4}{3\pi}|u|u}_{\text{resonant part}} + \underbrace{\sum_{m\neq 0} \frac{4(-1)^{m+1}}{\pi(2m-1)(2m+1)(2m+3)} |u|^{1-2m} u^{1+2m}.}_{\text{non-resonant part}}$$

cf. Masaki-Segata (arxiv '16 & '17) $\cdots$ NLKG with $|u|^{\frac{2}{d}}u$ in 2 & 3d.
Sunagawa ('06) $\cdots$ NLKG with cubic homogeneous polynomial in 1d.

Main Theorem 1 $\quad F(u) = g_0|u|^{\frac{5}{3}} + g_1|u|^{\frac{2}{3}}u + \sum_{n\neq 0,1} g_n|u|^{\frac{5}{3}-n}u^n.$

Assumption 1.1

F : homogeneous of degree $5/3$, $g_0 = 0$, $g_1 \in \mathbb{R}$ and for some $\eta > 0$,

$$\sum_{n\in\mathbb{Z}} |n|^{1+\eta}|g_n| < \infty.$$

Theorem 1

F satisfies the Assumption 1.1. Suppose $\delta \in (3/2, 5/3)$ so that $\delta - 3/2 < 2\eta$. Let $b \in (3/4, \delta/2)$. There exists $\varepsilon_0 = \varepsilon_0(b, \|g\|_{\mathrm{Lip}})$ with the following property: For any $u_+ \in H^{0,2} \cap \dot{H}^{-\delta}$ with $\|\widehat{u}_+\|_{L^\infty} < \varepsilon_0$, there exists $T > 0$ such that (NLS) admits a unique solution $u \in C([T,\infty); L^2(\mathbb{R}^3))$ satisfying

$$\sup_{t\in[T,\infty)} t^b\|u(t) - u_p(t)\|_{L^2} < \infty,$$

where $$u_p(t) = (2it)^{-\frac{3}{2}} e^{\frac{i|x|^2}{4t}} \widehat{u}_+\left(\frac{x}{2t}\right) \exp(-ig_1|\widehat{u}_+|^{\frac{2}{3}} \log t).$$

Weighted Sobolev space and homogeneous Sobolev space: For $s, m \in \mathbb{R}$,

$$H^{0,m} := \left\{\phi \in \mathcal{S}' \ : \ \|\phi\|_{H^{s,m}} = \|(1+|x|^2)^{\frac{m}{2}}\phi\|_{L^2} < \infty\right\},$$

$$\dot{H}^s := \left\{\phi \in \mathcal{S}' \ : \ \|\phi\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2} < \infty\right\}.$$

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where $$u_p(t)=(2it)^{-\frac{3}{2}}e^{\frac{i|x|^2}{4t}}\widehat{u}_+\left(\frac{x}{2t}\right)\exp(-ig_1|\widehat{u}_+|^{\frac{2}{3}}\log t).$$

Theorem 1 includes $F(u)=|\mathrm{Re}\,(u)|^{\frac{2}{3}}\mathrm{Re}\,(u)$ as an example.

$$g_n=\begin{cases}\dfrac{(-1)^{\frac{n-1}{2}}\Gamma(\frac{11}{6})\Gamma(\frac{3n-5}{6})}{\sqrt{\pi}\Gamma(-\frac{1}{3})\Gamma(\frac{3n+11}{6})} & (n:\text{odd}),\\ 0 & (n:\text{even}).\end{cases}\qquad g_n=O(|n|^{-\frac{8}{3}})\quad(|n|\to\infty).$$

Main Theorem 2 (concerning the second asymptotic profile)

Theorem 2

Under the same assumptions in Thm 1, the sol. obtained in Thm 1 satisfies

$$\sup_{t\in[T,\infty)} t^b \|u - u_p - \mathcal{V}\|_{L^\infty(t,\infty;L^2)\cap L^2(t,\infty;L^6)} < \infty$$

for any $b < \delta/2$*, where*

$$\mathcal{V}(t) = -\mathcal{F} \sum_{n\neq 0,1} \frac{g_n}{2(in)^{3/2}} \left(\frac{|\widehat{u}_+|^{\frac{5}{3}-n} \widehat{u}_+^n i^{-\frac{3}{2}n} e^{-int|\cdot|^2} e^{-ing_1|\widehat{u}_+|^{\frac{2}{3}}\log t}}{1 + in(n-1)t|\cdot|^2} \right) \left(\frac{\xi}{n}\right).$$

It holds that for any $b < \delta/2$*,*

$$\sup_{t\in[T,\infty)} t^b \|\mathcal{V}\|_{L^\infty(t,\infty;L^2)} < \infty, \qquad \sup_{t\in[T,\infty)} t^b \|\mathcal{V} - v_p\|_{L^2(t,\infty;L^6)} < \infty,$$

where $\quad v_p(t) := -i \sum_{n\neq 0,1} g_n \frac{1}{t^{-1} - i\frac{n-1}{n}\Delta} |u_p(t)|^{\frac{5}{3}-n} u_p(t)^n.$

By Hörmander-Mikhlin's theorem, we have

$$\|v_p\|_{L^2(t,\infty;L^6)} \leq Ct^{-\frac{1}{2}}.$$

However, we have no lower bound of decay of v_p.

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It holds that for any $b < \delta/2$,

$$\sup_{t\in[T,\infty)} t^b \|\mathcal{V}\|_{L^\infty(t,\infty;L^2)} < \infty, \qquad \sup_{t\in[T,\infty)} t^b \|\mathcal{V} - v_p\|_{L^2(t,\infty;L^6)} < \infty,$$

where $$v_p(t) := -i \sum_{n\neq 0,1} g_n \frac{1}{t^{-1} - i\frac{n-1}{n}\Delta} |u_p(t)|^{\frac{5}{3}-n} u_p(t)^n.$$

By Hörmander-Mikhlin's theorem, we have

we have to show $\rightarrow ct^{-\frac{3}{4}+\varepsilon} \leq \|v_p\|_{L^2(t,\infty;L^6)} \leq Ct^{-\frac{1}{2}}$.

However, we have no lower bound of decay of v_p.

Main Theorem 2 (concerning the second asymptotic profile)

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Under the same assumptions in Thm 1, the sol. obtained in Thm 1 satisfies

$$\sup_{t\in[T,\infty)} t^b \|u - u_p - \mathcal{V}\|_{L^\infty(t,\infty;L^2)\cap L^2(t,\infty;L^6)} < \infty$$

for any $b < \delta/2$, where

$$\mathcal{V}(t) = -\mathcal{F}\sum_{n\neq 0,1} \frac{g_n}{2(in)^{3/2}} \left(\frac{|\widehat{u}_+|^{\frac{5}{3}-n}\widehat{u}_+^n i^{-\frac{3}{2}n} e^{-int|\cdot|^2} e^{-ing_1|\widehat{u}_+|^{\frac{2}{3}}\log t}}{1 + in(n-1)t|\cdot|^2} \right)\left(\frac{\xi}{n}\right).$$

It holds that for any $b < \delta/2$,

$$\sup_{t\in[T,\infty)} t^b \|\mathcal{V}\|_{L^\infty(t,\infty;L^2)} < \infty, \qquad \sup_{t\in[T,\infty)} t^b \|\mathcal{V} - v_p\|_{L^2(t,\infty;L^6)} < \infty,$$

where $\quad v_p(t) := -i\sum_{n\neq 0,1} g_n \dfrac{1}{t^{-1} - i\frac{n-1}{n}\Delta} |u_p(t)|^{\frac{5}{3}-n} u_p(t)^n.$

Additionally, if $u_+ \in \dot{H}^{-2-\varepsilon}$ $(\varepsilon > 0)$, then

$$v_p(t) = -i\sum_{n\neq 0,1} \frac{g_n}{n(1-n)} \left|\frac{2t}{x}\right|^2 |u_p(t)|^{\frac{5}{3}-n} u_p(t)^n + o(t^{-1}) \text{ in } L^2.$$

cf. Moriyama-Tonegawa-Tsutsumi ('03).

Outline of the proof of Theorem 1

By Assumption 1.1,

$$F(u) = g_1|u|^{\frac{2}{3}}u + \sum_{n\neq 0,1} g_n|u|^{\frac{5}{3}-n}u^n = \mathcal{G}(u) + \mathcal{N}(u).$$

Let $M(t)\phi = e^{\frac{i|x|^2}{4t}}$, $(D(t)\phi)(x) = (2it)^{-\frac{d}{2}}\phi\left(\frac{x}{2t}\right)$,

$$u_p(t) = M(t)D(t)\widehat{w}(t), \quad \widehat{w}(t) = \widehat{u}_+ \exp(-ig_1|\widehat{u}_+|^{\frac{2}{3}}\log t).$$

We consider the following integral equation:

$$\begin{aligned} u(t) - u_p(t) = {} & i\int_t^\infty U(t-s)(F(u) - F(u_p))(s)ds + \mathcal{R}(t)\widehat{w} \\ & - i\int_t^\infty U(t-s)\mathcal{R}(t)\mathcal{G}(\widehat{w})(s)\frac{ds}{s} + i\int_t^\infty U(t-s)\mathcal{N}(u_p)(s)ds, \end{aligned}$$

where $\mathcal{R}(t) = M(t)D(t)\left(U\left(-\frac{1}{4t}\right) - 1\right)$.
For $R, T > 0$, $3/4 < b < \delta/2$, $\|v\|_{X_{T,b}} := \sup_{t\in[T,\infty)} t^b\|v(t)\|_{L^2}$,

$$X_{T,b,R} := \{v \in C([T,\infty); L^2(\mathbb{R}^3))\,;\, \|v - u_p\|_{X_{t,b}} < \infty\}.$$

Difference between $d = 1, 2$ and $d = 3$:

- Use of the end-point Strichartz estimate (peculiar in $d \geq 3$).
- Modification of the regularizing operator (Give an improvement in 2d).

Outline of the proof of Theorem 1

To estimate the term

$$i \int_t^\infty U(t-s)\mathcal{N}(u_p)(s)ds,$$

we use the following.

Regularizing operator

For $\psi \in \mathcal{S}$ and $\psi(0) = 1$, we define

$$K_\psi \phi := \psi\left(\frac{i\nabla}{|n|\sqrt{t}}\right)\phi = \mathcal{F}^{-1}\psi\left(\frac{\xi}{|n|\sqrt{t}}\right)\mathcal{F}\phi.$$

Let $s \in \mathbb{R}$, $0 \leq \theta \leq 2$ and assume $\nabla\psi(0) = 0$ if $1 < s \leq 2$. Then for any $n \neq 0$, we have

$$\|(K_\psi - 1)\phi\|_{\dot{H}^s} \leq Ct^{-\frac{\theta}{2}}|n|^{-\theta}\|\phi\|_{\dot{H}^{s+\theta}}$$

cf. Hayashi-Naumkin-Wang ('11) $\cdots$ Not depend on n.
Masaki-Miyazaki (arxiv '16) $\cdots 0 \leq \theta \leq 1$

$$\psi\left(\frac{\xi}{|n|t^{\frac{\sigma}{2}}}\right) \quad \begin{pmatrix} \sigma = 1 \ (d=1), \\ \sigma = \dfrac{2+\delta}{3} \ (d=2) \end{pmatrix}.$$

Outline of the proof of Theorem 1

Recall $u_p(t) = M(t)D(t)\widehat{w}(t), \quad \widehat{w}(t) = \widehat{u}_+ \exp(-ig_1|\widehat{u}_+|^{\frac{2}{3}} \log t)$, and

$$\mathcal{N}(u) = \sum_{n\neq 0,1} g_n |u|^{\frac{5}{3}-n} u^n.$$

By using $E(t) = e^{it|x|^2}$ and $\phi_n(t) = |\widehat{w}(t)|^{\frac{5}{3}-n}\widehat{w}(t)$,

$$\begin{aligned}
\mathcal{N}(u_p) &= \sum_{n\neq 0,1} g_n \left(\frac{1}{2t} D(t) i^{-\frac{3}{2}(n-1)} E^n(t)\phi_n(t) \right) \\
&= \sum_{n\neq 0,1} g_n \left(\frac{1}{2t} D(t) i^{-\frac{3}{2}(n-1)} E^n(t) K_\psi \phi_n(t) \right) \\
&\quad - \sum_{n\neq 0,1} g_n \left(\frac{1}{2t} D(t) i^{-\frac{3}{2}(n-1)} E^n(t) (K_\psi - 1)\phi_n(t) \right) \\
&= P + Q.
\end{aligned}$$

Thus we have the estimate for high-frequency part

$$\begin{aligned}
\|Q(t)\|_{L^2} &\leq Ct^{-1} \sum_{n\neq 0,1} |g_n| \|(K_\psi - 1)\phi_n(t)\|_{L^2} \\
&\leq Ct^{-1-\frac{\delta}{2}} \sum_{n\neq 0,1} |n|^{-\delta} |g_n| \|\phi_n(t)\|_{\dot{H}^\delta}.
\end{aligned}$$

Since $\delta > \frac{3}{2}$, we need to modify the regularizing estimate.

Outline of the proof of Theorem 1

Hayashi-Naumkin-Wang : $\mathcal{F}U(-s)D(s)E^{\rho}(s) = i^{\frac{3}{2}}E^{1-\frac{1}{\rho}}(s)U\left(\frac{\rho}{4s}\right)D\left(\frac{\rho}{2}\right)$.
We rewrite the low-frequency part

$$\int_t^\infty U(t-s)P(s)ds = U(t)\mathcal{F}^{-1}\int_t^\infty \mathcal{F}U(-s)P(s)ds,$$

and apply above identity to obtain

$$\begin{aligned}\mathcal{F}U(-s)P(s) &= \sum_{n\neq 0,1} i^{-\frac{3}{2}(n-1)}g_n\frac{1}{2s}\mathcal{F}U(-s)D(s)E^n(s)K_\psi\phi_n(s)\\ &= \sum_{n\neq 0,1} i^{-\frac{3}{2}(n-2)}g_n\frac{1}{2s}E^{1-\frac{1}{n}}(s)U\left(\frac{n}{4s}\right)D\left(\frac{n}{2}\right)K_\psi\phi_n(s).\end{aligned}$$

Since we have for $n\neq 0,1$, $E(t) = e^{it|x|^2}$

$$E^{1-\frac{1}{n}}(s) = A_n(s)\partial_s(sE^{1-\frac{1}{n}}(s)),\quad A_n(s) = \left(1+i\left(1-\frac{1}{n}s|x|^2\right)\right)^{-1}.$$

We perform integration by parts

$$\begin{aligned}&U(t)\mathcal{F}^{-1}\int_t^\infty \mathcal{F}U(-s)P(s)ds\\ &= \sum_{n\neq 0,1} i^{-\frac{3}{2}(n-2)}g_nU(t)\mathcal{F}^{-1}\int_t^\infty E^{1-\frac{1}{n}}(s)U\left(\frac{n}{4s}\right)D\left(\frac{n}{2}\right)K_\psi\phi_n(s)ds.\end{aligned}$$