An existence and uniqueness result for the Navier-Stokes type equations on the Heisenberg group

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2018 年 11 月 17 日 (土)

研究目的

本講演の目的

- ユークリッド空間（可換）上の Navier-Stokes 方程式の初期値問題を、ハイゼンベルグ群（非可換群）上の Navier-Stokes 方程式の初期値問題に置き換えて、可解性（存在、一意性）の解析を行う。

Navier-Stokes type equation on $\mathbb{H}^1$

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \tilde{\nabla}\pi = 0,\ g = (x,y,s) \in \mathbb{H}^1,\ t > 0, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g,0) = \boldsymbol{a}(g) \end{cases} \tag{0.1}$$

Theorem 1′ (O ‘19)

The following existence and uniqueness of local and global solutions of (0.1):

- If $\boldsymbol{a} \in J^4(\mathbb{H}^1)$, there exists a unique solution $\boldsymbol{u} \in C([0,T), J^4(\mathbb{H}^1))$ for $T > 0$. $(J^4(\mathbb{H}^1) = L^4(\mathbb{H}^1 \cong \mathbb{R}^3)^2,\ \mathrm{div}_R \boldsymbol{w} = 0)$ (0.2)
- Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^4} \leq \delta$, we can take $T = \infty$ in (0.2).

• Y. Oka, *An existence and uniqueness result for the Navier-Stokes type equations on the Heisenberg group*, J. Math. Anal. Appl. 473 (2019), no. 1, 382-407.

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ユークリッド空間上の Navier-Stokes 方程式の初期値問題の紹介

ユークリッド空間上の Navier-Stokes 方程式

非圧縮性粘性流体方程式

$$\begin{cases} \boldsymbol{u}_t - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = 0, & (x,y,z) \in \mathbb{R}^3,\ t > 0, \\ \mathrm{div}\,\boldsymbol{u} = 0, & (x,y,z) \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{a}, & (x,y,z) \in \mathbb{R}^3, \end{cases} \tag{0.3}$$

← 運動方程式
← 非圧縮条件
← 初期条件

- $\boldsymbol{u} = (u_1, u_2, u_3)$(未知), π: 圧力項 (未知), $\boldsymbol{u}_t = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_3}{\partial t}\right)$,
- $\Delta u_i = \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 u_i}{\partial z^2}$ $(i = 1,2,3)$ (粘性項), $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
- $\mathrm{div}\,\boldsymbol{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$, $\boldsymbol{u}\cdot\nabla u_i = u_1\frac{\partial u_i}{\partial x} + u_2\frac{\partial u_i}{\partial y} + u_3\frac{\partial u_i}{\partial z}$ $(i = 1,2,3)$.

ユークリッド空間上の Navier-Stokes 方程式

(0.3) を成分で書き表すと,

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} - \dfrac{\partial^2 u_1}{\partial x^2} - \dfrac{\partial^2 u_1}{\partial y^2} - \dfrac{\partial^2 u_1}{\partial z^2} + u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y} + u_3 \dfrac{\partial u_1}{\partial z} + \dfrac{\partial \pi}{\partial x} = 0, \\
\dfrac{\partial u_2}{\partial t} - \dfrac{\partial^2 u_2}{\partial x^2} - \dfrac{\partial^2 u_2}{\partial y^2} - \dfrac{\partial^2 u_2}{\partial z^2} + u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y} + u_3 \dfrac{\partial u_2}{\partial z} + \dfrac{\partial \pi}{\partial y} = 0, \\
\dfrac{\partial u_3}{\partial t} - \dfrac{\partial^2 u_3}{\partial x^2} - \dfrac{\partial^2 u_3}{\partial y^2} - \dfrac{\partial^2 u_3}{\partial z^2} + u_1 \dfrac{\partial u_3}{\partial x} + u_2 \dfrac{\partial u_3}{\partial y} + u_3 \dfrac{\partial u_3}{\partial z} + \dfrac{\partial \pi}{\partial z} = 0, \\
\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} + \dfrac{\partial u_3}{\partial z} = 0, \\
u_1(x, y, z, 0) = a_1(x, y, z),\ u_2(x, y, z, 0) = a_2(x, y, z), \\
u_3(x, y, z, 0) = a_3(x, y, z).
\end{cases}
$$

ユークリッド空間上の Navier-Stokes 方程式

- 水などの非圧縮性 ($\mathrm{div} u = 0$) の流体の運動方程式（膨張や収縮なし）
- 粘性係数 $\mu = 1$ の場合（μ は Δ の前に付く係数）
- $\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ は速度ベクトル場 $\boldsymbol{u}$ の流れに沿った時間微分を表す.
- 非圧縮性粘性流体方程式を Navier-Stokes 方程式という.

ユークリッド空間上の Navier-Stokes 方程式

- $J^p(\mathbb{R}^n) = \{\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in L^p(\mathbb{R}^n)^n \mid \mathrm{div}\boldsymbol{a} = 0\}$ (Solenoidal space)

Theorem ($\mathbb{R}^m$-Navier-Stokes equations) (T. Kato '84, Y. Giga '86)

Let $m \geq 2$. Then the following existence and uniqueness of local and global solutions of (0.3):

· If $\boldsymbol{a} \in J^m(\mathbb{R}^m)$, there exists a unique solution $\boldsymbol{u} \in C([0,T), J^m(\mathbb{R}^m))$ for $T > 0$. (0.4)

· Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^m} \leq \delta$, we can take $T = \infty$ in (0.4).

Theorem (2D-Navier-Stokes equations)

The following existence and uniqueness of local and global solutions of (0.3):

- If $\boldsymbol{a} \in J^2(\mathbb{R}^2)$, there exists a unique solution $\boldsymbol{u} \in C([0,T), J^2(\mathbb{R}^2))$ for $T > 0$. ($J^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)^2,\ \mathrm{div}\,\boldsymbol{w} = 0$.) (0.5)
- Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^2} \leq \delta$, we can take $T = \infty$ in (0.5).

Theorem (3D-Navier-Stokes equations)

The following existence and uniqueness of local and global solutions of (0.3):

- If $\boldsymbol{a} \in J^3(\mathbb{R}^3)$, there exists a unique solution $\boldsymbol{u} \in C([0,T), J^3(\mathbb{R}^3))$ for $T > 0$. ($J^3(\mathbb{R}^3) = L^3(\mathbb{R}^3)^3,\ \mathrm{div}\,\boldsymbol{w} = 0$) (0.6)
- Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^3} \leq \delta$, we can take $T = \infty$ in (0.6).

ハイゼンベルグ群とは？

ハイゼンベルグ群の定義と性質

- $g = (x, y, s),\ g' = (x', y', s') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{2d+1}$
- We consider $\mathbb{R}^{2d+1}$ with the group law $\cdot$ defined by

$$\begin{aligned} g \cdot g' &= (x, y, s) \cdot (x', y', s') \\ &= (x + x', y + y', s + s' + 1/2(x' \cdot y - x \cdot y')), \end{aligned} \tag{0.7}$$

  where $x \cdot y = \sum_{j=1}^{d} x_j y_j$.
- The group $(\mathbb{R}^{2d+1}, \cdot\ )$ with respect to the group law $\cdot$ defined by (0.7) is called the Heisenberg group and denoted by $\mathbb{H}^d$.
- $e = (0, 0, 0),\ g^{-1} = (-x, -y, -s)$

ハイゼンベルグ群の定義と性質

- The left-invariant vector fields ($[Xf](g' \cdot g) = X[f(g' \cdot g)]$) in the Heisenberg group $\mathbb{H}^d$ are represented by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial s}, \ X_{d+j} = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial s}, \ X_{2d+1} = \frac{\partial}{\partial s}$$

  for $j = 1, 2, \cdots, d$.
- The right-invariant vector fields ($[\tilde{X} f](g \cdot g') = \tilde{X}[f(g \cdot g')]$) in $\mathbb{H}^d$ are represented by

$$\tilde{X}_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial s}, \ \tilde{X}_{d+j} = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial s}, \ \tilde{X}_{2d+1} = \frac{\partial}{\partial s}$$

  for $j = 1, 2, \cdots, d$.
- These make a basis for the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^d$.

ハイゼンベルグ群の定義と性質

- $\Gamma_j = \{(\theta e_j, 0, 0),\ \theta \in \mathbb{R}\}, \Gamma_{d+j} = \{(0, \theta e_j, 0),\ \theta \in \mathbb{R}\},$
  $\Gamma_{2d+1} = \{(0, 0, \theta),\ \theta \in \mathbb{R}\}, j = 1, \cdots, d.$
- $X_j f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((x,y,s)\cdot(\theta e_j, 0, 0))$
  $X_{d+j} f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((x,y,s)\cdot(0, \theta e_j, 0))$
  $X_{2d+1} f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((x,y,s)\cdot(0, 0, \theta))$
- $\tilde{X}_j f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((\theta e_j, 0, 0)\cdot(x,y,s))$
  $\tilde{X}_{d+j} f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((0, \theta e_j, 0)\cdot(x,y,s))$
  $\tilde{X}_{2d+1} f(x,y,s) = \frac{d}{d\theta}|_{\theta=0} f((0, 0, \theta)\cdot(x,y,s))$

ハイゼンベルグ群の定義と性質

- (C.C.R. 正準交換関係)
$$\begin{cases}[X_j, X_{d+j}] = -X_{2d+1}, \\ 0 \quad (Others), \end{cases} \quad (j = 1, \cdots, d) \qquad \begin{cases}[X_1, X_2] = -X_3, \\ 0 \quad (Others). \end{cases} \quad (d = 1)$$

- 原点では $X_1 = \dfrac{\partial}{\partial x}, X_2 = \dfrac{\partial}{\partial y}, X_3 = \dfrac{\partial}{\partial s}$ $(d = 1)$.

- $\mathbb{H} \cong \mathbb{R}^3$ のリー環を $\mathfrak{h}$ とするとき, $V_1 = \{X_1, X_2\}$, $V_2 = \{X_3\}$ とおくと, $\mathfrak{h} = V_1 \oplus V_2$, $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$, $\mathrm{Lie}\{X_1, X_2\} = \mathrm{span}\{X_1, X_2, [X_1, X_2]\}$ より,
$$\mathrm{rank}\{\mathrm{Lie}\{X_1, X_2\}(g)\} = 3, \text{ for any } g \in \mathbb{H}.$$
$$(\text{Hörmander's condition})$$

ハイゼンベルグ群の定義と性質

- 2 step stratified Lie group.
- The induced subriemannian geometry is step 2. (2 step ベキ零リー群)
- この正準交換関係 (C.C.R.) がハイゼンベルグ群の名前の由来になっている。
（詳しくは、R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications, (2002), AMS）
- 最も可換に近い非可換群　(By S. Thangavelu).
- 多くの文献が存在する. (次のページ)

ハイゼンベルグ群の定義と性質

ハイゼンベルグ群の参考文献

- G. B. Folland, Harmonic Analysis in Phase Space, *Princeton University Press. Princeton, N.J.*, (1989).
- S. Thangavelu, An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups , *Birkhäuser, Boston*, (2004).
- E. M. Stein, Harmonic Analysis, *Princeton University Press.* Princeton, N.J., (1993). (12 章と 13 章)
- N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and geometry on groups, *Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge,* (1992).

ハイゼンベルグ群の定義と性質

- $\nabla = (X_1, X_2, \cdots, X_{2d})$, $\tilde{\nabla} = (\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_{2d})$
- $\mathrm{div}\boldsymbol{a} = X_1 a_1 + \cdots + X_{2d} a_{2d}$, $\mathrm{div}_R \boldsymbol{a} = \tilde{X}_1 a_1 + \cdots + \tilde{X}_{2d} a_{2d}$
- The sub-Laplacian $\mathcal{L}$ and $\mathcal{L}_R$ on $\mathbb{H}^d$ are defined by

$$\mathcal{L} = -\sum_{j=1}^{2d} X_j^2$$

and

$$\mathcal{L}_R = -\sum_{j=1}^{2d} \tilde{X}_j^2,$$

respectively. Thanks to Hörmander's result, the sublaplacian $\mathcal{L}$ and $\mathcal{L}_R$ are subelliptic. ($\mathcal{L}u = 0 \Rightarrow u \in C^\infty$)

ハイゼンベルグ群の定義と性質

- The Heisenberg group $\mathbb{H}^d$ is a locally compact Hausdorff group and its Haar measure is the Lebesgue measure $dxdydt$.
- $\delta_\lambda(x, y, s) = (\lambda x, \lambda y, \lambda^2 s)$, $\quad (x, y, s) \in \mathbb{H}^d$, $\lambda > 0$.
- The homogeneous dimension N of $\mathbb{H}^d$ is given by $N = 2d + 2$.

$$(\text{If } \mathfrak{h} = V_1 \oplus V_2, \text{ then } N = \sum_{i=1}^{2} i \dim V_i)$$

- $\displaystyle\int_{\mathbb{H}^d} f(\delta_\lambda(g))dg = \lambda^{-N} \int_{\mathbb{H}^d} f(g)dg.$
- A family $\{\delta_\lambda\}_{\lambda>0}$ is an automorphism of $\mathbb{H}^d$.
- The distance function (Korányi norm) ρ defined by $\rho(g) = ((x^2 + y^2)^2 + s^2)^{\frac{1}{4}}$ for $g = (x, y, s) \in \mathbb{H}^d$ satisfies

$$\rho(\lambda x, \lambda y, \lambda^2 s) = \lambda\rho(x, y, s), \ \lambda > 0.$$

ハイゼンベルグ群の定義と性質

Let f and h be suitable functions. Then the convolution $f * h$ of f with h on $\mathbb{H}^d$ is defined by

$$(f * h)(g) = \int_{\mathbb{H}^d} f(g')h(g'^{-1} \cdot g)dg' = \int_{\mathbb{H}^d} f(g \cdot g'^{-1})h(g')dg'.$$

The convolution $*$ is non commutative ($f * h \neq h * f$). The relationship between the invariant vector fields and the convolution is

$$X_i(f * h)(g) = (f * X_i h)(g) \text{ and } \tilde{X}_i(f * h)(g) = (\tilde{X}_i f * h)(g).$$

ハイゼンベルグ群の定義と性質

- $L^p(\mathbb{H}^d) = \left\{ f \mid \left(\int_{\mathbb{H}^d} |f(g)|^p dg \right)^{\frac{1}{p}} < \infty \right\}$, $\|f\|_p = \left(\int_{\mathbb{H}^d} |f(g)|^p dg \right)^{\frac{1}{p}}$
- $L^p(\mathbb{H}^d)^{2d} = \{ \boldsymbol{u} = (u_1, \cdots, u_{2d}) \mid u_j \in L^p(\mathbb{H}^d) \}$, $\|\boldsymbol{u}\|_p = \sum_{j=1}^{2d} \|u_j\|_p$
- $(2d \times 2d)$-matrix $\nabla \boldsymbol{u}$ is defined by $(X_k u_j)_{\substack{1 \le j \le 2d \\ 1 \le k \le 2d}}$ and $\nabla \boldsymbol{u} \in L^p(\mathbb{H}^d)^{2d \times 2d}$ means $X_k u_j \in L^p(\mathbb{H}^d)$.
- $\|\nabla \boldsymbol{u}\|_N = \sum_{j,k=1}^{2d} \|X_k u_j\|_N$.

$\mathbb{H}^d$ のサブラプラシアンに付随する熱核

Let h_t be the heat kernel associated to $\mathcal{L}$,

$$e^{-t\mathcal{L}}f(g) = (f * h_t)(g) = \int_{\mathbb{H}^d} f(g')h_t(g'^{-1} \cdot g)dg'.$$

Proposition 1 (G. B. Folland '75)

Let h_t be the heat kernel associated to $\mathcal{L}$. Then the following properties hold:

1. $h_t(g) \geq 0$,
2. $\displaystyle\int_{\mathbb{H}^d} h_t(g)dg = 1$,
3. $h_t(g) = h_t(g^{-1})$,
4. $(\partial/\partial t + \mathcal{L})\, h_t(g) = 0$ and
5. $h_{r^2 t}(rx, ry, r^2 s) = r^{-N} h_t(x, y, s)$, $r > 0$, $(x, y, s) \in \mathbb{H}^d$.

$\mathbb{H}^d$ のサブラプラシアンに付随する熱核

The explicit representation of the heat kernel h_t is known as follows.

Proposition 2 (A. Hulanicki, '76, B. Gaveau '77)

The heat kernel h_t associated to $\mathcal{L}$ is given by

$$h_t(g) = (2\pi)^{-1}(4\pi)^{-d}\int_{\mathbb{R}}\left(\frac{\lambda}{\sinh\lambda t}\right)^d e^{-\frac{\lambda\coth\lambda t}{4}(|x|^2+|y|^2)}e^{-i\lambda s}\,d\lambda,$$

for $g=(x,y,s)\in\mathbb{H}^d$.

$\mathbb{H}^d$ のサブラプラシアンに付随する熱核

With respect to the estimate of the heat kernel h_t associated to $\mathcal{L}$, the following is known.

Proposition 3 (D. S. Jerison and A. Sanchez-Calle '86)

Let $h_t(g)$ be the heat kernel associated to $\mathcal{L}$. Then there exist positive constants C_1 and $C_{l,l}$ depending $\mathcal{L}$ such that

$$|\partial_t^\alpha X_I h_t(g)| \le C_{I,\alpha} t^{-\alpha-\frac{|I|}{2}-\frac{N}{2}} e^{-\frac{c_1 \rho(g)^2}{t}},$$

where $I = (i_1, \cdots, i_m)$ with $|I| = m$ and $X_I = X_{i_1} X_{i_2} \cdots X_{i_m}$.

$L^p - L^q$ estimate on $\mathbb{H}^d$

Proposition 4 ($L^p - L^q$ estimate on $\mathbb{H}^d$)

Let $N = 2d + 2$ and assume $1 \leq q < p \leq \infty$. Then there exists a positive constant C such that for any $\varphi \in L^q(\mathbb{H}^d)$,

$$\|e^{-t\mathcal{L}}\varphi\|_p \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_q.$$

※熱核の評価とヘルダーの不等式で証明できる.

$\mathbb{H}^d$ 上の偏微分方程式の先行結果

$\mathbb{H}^d$ 上の偏微分方程式の先行結果

- $\mathcal{L}u = f$
  - · N. Garofalo and E. Lanconelli ('92)
  - · N. Garofalo and D. Vassilev ('02)
    (Yamabe type equation、H 型群)
- $u_t + \mathcal{L}u = f$
  - · G. B. Folland ('75)　(熱核の性質)
  - · A. Hulanicki, ('76), B. Gaveau ('77)
    (ハイゼンベルグ群の熱核の積分表示)
  - · D. S. Jerison and A. Sanchez-Calle ('86)　(熱核の評価)
  - · Q. Yang and F. Zhu ('08)　(H 型群の熱核の明示表示)
  - · Y. Oka ('18)　(半線形熱方程式の局所適切性、H 型群)
- $iu_t + \mathcal{L}u = f$, $u_{tt} + \mathcal{L}u = f$
  - · H, Bahouri, P. Gerard and C. -J. Xu ('00)
    (分散評価、Strichartz 評価、ハイゼンベルグ群)
  - · M. Hierro ('05)　(分散評価、Strichartz 評価、H 型群)

主結果

Navier-Stokes type equation on $\mathbb{H}^d$

We consider the following Cauchy problem of the Navier-Stokes type equations,

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \tilde{\nabla}\pi = 0, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g, 0) = \boldsymbol{a}(g) \end{cases} \tag{0.8}$$

for $g \in \mathbb{H}^d$ and $t > 0$, limited to Lebesgue spaces on 2-step stratified Lie groups, especially Heisenberg group, instead of the Euclidean spaces.

Navier-Stokes type equation on $\mathbb{H}^1$

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \tilde{\nabla}\pi = 0, \ g = (x, y, s) \in \mathbb{H}^1, \ t > 0, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g, 0) = \boldsymbol{a}(g) \end{cases} \tag{0.9}$$

- $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{u}_t = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right)$, π 未知,
- $\mathcal{L}u_i = -\frac{\partial^2 u_i}{\partial x^2} - \frac{\partial^2 u_i}{\partial y^2} - \frac{1}{4}(x^2 + y^2)\frac{\partial^2 u_i}{\partial s^2} - \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\frac{\partial u_i}{\partial s}$, $(i = 1, 2)$
- $\nabla = \left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial s}, \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial s}\right)$, $\tilde{\nabla} = \left(\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial s}, \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial s}\right)$

Navier-Stokes type equation on $\mathbb{H}^1$

- $\dfrac{\partial u_1}{\partial x} - \dfrac{1}{2} y \dfrac{\partial u_1}{\partial s} + \dfrac{\partial u_2}{\partial y} + \dfrac{1}{2} x \dfrac{\partial u_2}{\partial s} = 0$
- $(\boldsymbol{u} \cdot \nabla) u_i = u_1 \dfrac{\partial u_i}{\partial x} + u_2 \dfrac{\partial u_i}{\partial y} + \dfrac{1}{2} (y u_1 - x u_2) \dfrac{\partial u_i}{\partial s}$
  $(i = 1, 2)$

ヘルムホルツ型分解

Let $\boldsymbol{b} = (b_1, \cdots, b_{2d}) \in L^p(\mathbb{H}^d)^{2d}$, $1 < p < \infty$, and set

$$\boldsymbol{Pb} = (P_1\boldsymbol{b}, \cdots, P_{2d}\boldsymbol{b}), \; P_j\boldsymbol{b} = b_j + \sum_{k=1}^{2d} R_j \bar{R}_k b_k$$

and

$$Q\boldsymbol{b} = -\sum_{i=1}^{2d} \mathcal{L}_R^{-1} \tilde{X}_i b_i,$$

where $R_l = \tilde{X}_l \mathcal{L}_R^{-\frac{1}{2}}$, $\bar{R}_l = \mathcal{L}_R^{-\frac{1}{2}} \tilde{X}_l$, $l = 1, \cdots, 2d$ and

$$\mathcal{L}_R^{-\frac{1}{2}} f(g) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2 \mathcal{L}_R} f(g)\, dt$$

by the spectral theorem.

ヘルムホルツ型分解

Then we decompose $\boldsymbol{b}$ by

$$\boldsymbol{b} = \boldsymbol{P}\boldsymbol{b} + \tilde{\nabla} Q\boldsymbol{b}.$$

Indeed, we have

$$\begin{aligned}
P_j\boldsymbol{b} + \tilde{X}_j Q\boldsymbol{b} &= \overbrace{b_j + \sum_{k=1}^{2d} R_j \bar{R}_k b_k}^{P_j\boldsymbol{b}} \overbrace{- \tilde{X}_j \sum_{i=1}^{2d} \mathcal{L}_R^{-1} \tilde{X}_i b_i}^{\tilde{X}_j Q\boldsymbol{b}} \\
&= b_j + \tilde{X}_j \sum_{k=1}^{2d} \mathcal{L}_R^{-\frac{1}{2}} \mathcal{L}_R^{-\frac{1}{2}} \tilde{X}_k b_k - \tilde{X}_j \sum_{i=1}^{2d} \mathcal{L}_R^{-1} \tilde{X}_i b_i \\
&= b_j.
\end{aligned}$$

ヘルムホルツ型分解

Lemma 1 (T. Coulhon et al '96)

Let $R_i = \tilde{X}_i \mathcal{L}_R^{-\frac{1}{2}}$ and $\bar{R}_i = \mathcal{L}_R^{-\frac{1}{2}} \tilde{X}_i$, $i = 1, 2, \cdots, 2d$. Then for any $p \in (1, \infty)$, there exists a positive constant C_p such that for any d, we have

$$\|R_i f\|_p \leq C_p \|f\|_p \text{ and } \|\bar{R}_i f\|_p \leq C_p \|f\|_p$$

for any $f \in L^p(\mathbb{H}^d)$.

By Lemma 1 above, the operators R and $\bar{R}$ are L^p-bounded. Hence we can see

$$\|\boldsymbol{Pb}\|_p \leq C \|\boldsymbol{b}\|_p. \tag{0.10}$$

ヘルムホルツ型分解

$$\mathrm{div}_R \boldsymbol{Pb} = 0 \tag{0.11}$$

Indeed, we have

$$\begin{aligned}
\tilde{X}_j P_j \boldsymbol{b} &= \tilde{X}_j b_j + \tilde{X}_j \sum_{k=1}^{2d} R_j \bar{R}_k b_k \\
&= \tilde{X}_j b_j + \tilde{X}_j^2 \sum_{k=1}^{2d} \mathcal{L}_R^{-\frac{1}{2}} \mathcal{L}_R^{-\frac{1}{2}} \tilde{X}_k b_k \\
&= \tilde{X}_j b_j - (-\tilde{X}_j^2) \sum_{k=1}^{2d} \mathcal{L}_R^{-1} \tilde{X}_k b_k .
\end{aligned}$$

Therefore we obtain

$$\mathrm{div}_R \boldsymbol{Pb} = \sum_{j=1}^{2d} \tilde{X}_j P_j \boldsymbol{b} = \sum_{j=1}^{2d} \tilde{X}_j b_j - \sum_{k=1}^{2d} \tilde{X}_k b_k = 0. \quad \square$$

Stokes type equation on $\mathbb{H}^d$

If $\pi = -Q(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, then

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \tilde{\nabla}\pi = 0, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g, 0) = \boldsymbol{a}(g) \end{cases}$$

becomes

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} + \boldsymbol{P}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \tilde{\nabla} Q(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \tilde{\nabla} Q(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = 0, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g, 0) = \boldsymbol{a}(g) \end{cases} .$$

Stokes type equation on $\mathbb{H}^d$

So we obtain

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} = -\boldsymbol{P}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g,0) = \boldsymbol{a}(g). \end{cases} \tag{0.12}$$

(Stokes equation)

$$\begin{cases} \boldsymbol{u}_t + \mathcal{L}\boldsymbol{u} = \boldsymbol{P}\boldsymbol{f}, \\ \mathrm{div}_R \boldsymbol{u} = 0, \\ \boldsymbol{u}(g,0) = \boldsymbol{a}(g). \end{cases} \tag{0.13}$$

Stokes type equation on $\mathbb{H}^d$(非斉次熱方程式)

Lemma 2 (非斉次熱方程式)

Assume $1 < p, q < \infty$ and $T > 0$ or $T = \infty$ and

- $u_0 \in L^p(\mathbb{H}^d)$ and $f(t) \in C((0,T), L^p(\mathbb{H}^d)) \cap L^q((0,T), L^p(\mathbb{H}^d))$.

If we set

$$u(g,t) = S(t)u_0(g) + \int_0^t S(t-\sigma)f(\sigma)d\sigma,$$

where $S(t)F = F * h_t$, then u satisfies the following conditions:

1. $u(g,t) \in C([0,T), L^p(\mathbb{H}^d))$,
2. $u(g,t)$ is a solution of $u_t + \mathcal{L}u = f$ in the sence of distributions and
3. $\lim_{t \to 0+} \|u(\cdot,t) - u_0\|_p = 0$.

Stokes type equation on $\mathbb{H}^d$

- $\boldsymbol{a} \in J^p(\mathbb{H}^d) = \{\boldsymbol{a} = (a_1, \cdots, a_{2d}) \in L^p(\mathbb{H}^d)^{2d},\ \mathrm{div}_R \boldsymbol{a} = 0\}$

Proposition 1 (Stokes equation)

Let $1 < p, q < \infty$ and $T > 0$ or $T = \infty$. For $\boldsymbol{a} \in J^p(\mathbb{H}^d)$ and

$$\boldsymbol{f}(t) \in C((0,T), L^p(\mathbb{H}^d)^{2d}) \cap L^q((0,T), L^p(\mathbb{H}^d)^{2d}).$$

If we set

$$\boldsymbol{u}(t) = S(t)\boldsymbol{a} + \int_0^t S(t-\sigma)\boldsymbol{Pf}(\sigma)d\sigma,$$

where $S(t)\boldsymbol{F} = (F_1 * h_t, F_2 * h_t, \cdots, F_{2d} * h_t)$, then $\boldsymbol{u}(t)$ is a solution of Stokes equations (0.13) satisfying the following conditions:

1. $\boldsymbol{u}(t) \in C([0,T), L^p(\mathbb{H}^d)^{2d})$,
2. $\mathrm{div}_R \boldsymbol{u}(t) = 0$ and
3. $\lim_{t \to 0+} \|\boldsymbol{u}(t) - \boldsymbol{a}\|_p = 0$.

Stokes type equation on $\mathbb{H}^d$

<Proposition 1 の証明>

Assume that

$$\boldsymbol{a} \in J^p(\mathbb{H}^d) = \{\boldsymbol{a} = (a_1, \cdots, a_{2d}) \in L^p(\mathbb{H}^d)^{2d},\ \mathrm{div}_R \boldsymbol{a} = 0\}$$

and

$$\boldsymbol{f}(t) \in C((0,T), L^p(\mathbb{H}^d)^{2d}) \cap L^q((0,T), L^p(\mathbb{H}^d)^{2d}).$$

By (0.10) (L^p boundedness of $\boldsymbol{P}$), we also see $\boldsymbol{Pf}(t) \in C((0,T), L^p(\mathbb{H}^d)^{2d}) \cap L^q((0,T), L^p(\mathbb{H}^d)^{2d})$. Now we set

$$u_j(g,t) = S(t)a_j(g) + \int_0^t S(t-\sigma)P_j\boldsymbol{f}(\sigma)d\sigma,\ j = 1, \cdots, 2d.$$

Then by Lemma 2 (非斉次熱方程式), $u_j(g,t)$ satisfies

$$u_j(g,t) \in C([0,T), L^p(\mathbb{H}^d)) \text{ and } \lim_{t\to 0+} \|u_j(\cdot,t) - a_j\|_p = 0.$$

Stokes type equation on $\mathbb{H}^d$

Moreover by Lemma 2, we have

$$\left\langle u_j, \partial_t \varphi_j - \mathcal{L}\varphi_j \right\rangle + \left\langle f_j + \sum_{k=1}^{2d} R_j \bar{R}_k f_k, \varphi_j \right\rangle = 0,\ j = 1, \cdots, 2d$$

for any $\varphi_j \in C_0^\infty(\mathbb{H}^d \times (0, T))$. Finally, by $\mathrm{div}_R \boldsymbol{a} = 0$ and $\mathrm{div}_R \boldsymbol{Pf}(t) = 0$ ((0.7)), we implies $\mathrm{div}_R \boldsymbol{u} = 0$. Indeed, since

$$\tilde{X}_j S(t) a_j(g) = (\tilde{X}_j a_j * h_t)(g),$$

we can see $\mathrm{div}_R S(t)\boldsymbol{a}(g) = 0$. Similarly, from $\mathrm{div}_R \boldsymbol{Pf}(t) = 0$, we obtain

$$\mathrm{div}_R \int_0^t S(t-\sigma)\boldsymbol{Pf}(\sigma)d\sigma = 0.$$

Therefore $u_j(g, t)$ $(j = 1, \cdots, 2d)$ is a solution of (0.13).

主結果 Local

We consider the following integral equation

$$\boldsymbol{u}(t) = S(t)\boldsymbol{a} - \int_0^t S(t-\sigma)\boldsymbol{P}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}(\sigma)d\sigma. \tag{0.14}$$

主結果 Local

Main Theorem 1 (O '19)

Let $N = 2d + 2 < p < \infty$. Then for an initial value $\boldsymbol{a} \in J^N(\mathbb{H}^d) = \{\boldsymbol{a} = (a_1, \cdots, a_{2d}) \in L^N(\mathbb{H}^d)^{2d},\ \mathrm{div}_R \boldsymbol{a} = 0\}$, there exists a positive constant T and the following holds:

- (Existence) The integral equation (0.14) has a solution $\boldsymbol{u}(t)$ satisfying the following conditions:

$$\cdot\ \boldsymbol{u}(t) \in C([0, T), L^N(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{a}\|_N = 0,$$

$$\cdot\ \mathrm{div}_R \boldsymbol{u}(t) = 0, \tag{0.15}$$

$$\cdot\ \boldsymbol{u}(t) \in C((0, T), L^p(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}-\frac{N}{2p}} \|\boldsymbol{u}(\sigma)\|_p = 0 \tag{0.16}$$

$$\cdot\ \nabla \boldsymbol{u}(t) \in C((0, T), L^N(\mathbb{H}^d)^{2d\times 2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}} \|\nabla \boldsymbol{u}\|_N = 0. \tag{0.17}$$

主結果 Local

- (Uniqueness) If $\boldsymbol{v}(t)$ satisfies (0.14), (0.15), (0.16) and (0.17), then $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for $0 \leq t < T$.

主結果 Global

Main Theorem 2 (O'19)

Let $N = 2d + 2 < p < \infty$.

- (Existence) If an initial value $\boldsymbol{a} \in J^N(\mathbb{H}^d)$ satisfies $\|\boldsymbol{a}\|_N \le C_{N,p}$ for a positive constant $C_{N,p}$, the integral equation (0.14) has a solution $\boldsymbol{u}(t)$ satisfying the following conditions.

$$\cdot\ \boldsymbol{u}(t) \in C([0,\infty), L^N(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{a}\|_N = 0,$$

$$\cdot\ \mathrm{div}_R\, \boldsymbol{u}(t) = 0, \tag{0.18}$$

$$\cdot\ \boldsymbol{u}(t) \in C((0,\infty), L^p(\mathbb{H}^d)^{2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}-\frac{N}{2p}} \|\boldsymbol{u}(\sigma)\|_p = 0, \tag{0.19}$$

$$\cdot\ \nabla\boldsymbol{u}(t) \in C((0,\infty), L^N(\mathbb{H}^d)^{2d\times 2d}),\ \lim_{t\to 0+} \sup_{0<\sigma<t} \sigma^{\frac{1}{2}} \|\nabla\boldsymbol{u}\|_N = 0 \tag{0.20}$$

$\cdot$ there exists a positive constant C such that

$$\|\boldsymbol{u}(t)\|_p \le Ct^{-\frac{1}{2}-\frac{N}{2p}},\ \|\nabla\boldsymbol{u}(t)\|_N \le Ct^{-\frac{1}{2}}.$$

主結果 Global

- (Uniqueness) If $\boldsymbol{v}(t)$ satisfies the integral equation (0.14), (0.18), (0.19) and (0.20). Then $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for $0 \leq t < \infty$.

$\mathbb{R}^2$ と $\mathbb{H}$ の結果の比較　※ $\mathbb{H}^1 \cong \mathbb{R}^3$

Theorem 1 (2D-Navier-Stokes equations)

The following existence and uniqueness of local and global solutions of (0.3):

- If $\boldsymbol{a} \in J^2(\mathbb{R}^2)$, there exists a unique solution $\boldsymbol{u} \in C([0,T), J^2(\mathbb{R}^2))$ for $T > 0$. $(J^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)^2,\ \mathrm{div}\boldsymbol{w} = 0)$ (0.21)
- Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^2} \leq \delta$, we can take $T = \infty$ in (0.21).

Theorem 1′ (O '19)

The following existence and uniqueness of local and global solutions of (0.9):

- If $\boldsymbol{a} \in J^4(\mathbb{H}^1)$, there exists a unique solution$\boldsymbol{u} \in C([0,T), J^4(\mathbb{H}^1))$ for $T > 0$. $(J^4(\mathbb{H}^1) = L^4(\mathbb{H}^1 \cong \mathbb{R}^3)^2,\ \mathrm{div}_R \boldsymbol{w} = 0)$ (0.22)
- Moreover if there exists a constant δ such that $\|\boldsymbol{a}\|_{L^4} \leq \delta$, we can take $T = \infty$ in (0.22).

Definition (Stratified Lie group)

A straitified Lie group (or Carnot group) $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. a direct sum decomposition

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$$

such that

$$\begin{cases} [V_1, V_{i-1}] = V_i, \ 2 \le i \le r \\ [V_1, V_r] = \{0\}. \end{cases}$$

• The Euclidean group $\mathbb{E} = (\mathbb{R}^d, +, \delta_\lambda)$ is a stratified Lie group of step 1.

$$\delta_\lambda(x) = \lambda x, \ \lambda > 0.$$

[Bon] A. Bonfiglioli, E. Lanconelli and E. Uguzzoni, Stratified Lie groups and Potential Theory for their Sub-Laplacians, *Splinger-Verlag Berlin Heidelberg* (2007).

2D-Navier-Stokes equations (1 step straitified Lie group)

$\boldsymbol{u} \in C([0,T), J^{2}(\mathbb{R}^2))$ $(J^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)^2,\ \mathrm{div}\,\boldsymbol{w} = 0)$

- $V_1 = \{\partial/\partial x, \partial/\partial y\}$, $[V_1, V_1] = 0$

$\mathbb{H}^1$-Navier-Stokes equations (2 step straitefied Lie group)

$\boldsymbol{u} \in C([0,T), J^{4}(\mathbb{H}^1))$ $(J^4(\mathbb{H}^1) = L^4(\mathbb{H}^1 \cong \mathbb{R}^3)^2,\ \mathrm{div}_R\,\boldsymbol{w} = 0)$

- $V_1 = \{X_1, X_2\}$, $V_2 = \{X_3\}$ とおくと, $\mathfrak{h} = V_1 \oplus V_2$, $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$.
- $\mathrm{rank}\{\mathrm{Lie}\{X_1, X_2\}(g)\} = 3$, for any $g \in \mathbb{H}$.

3D-Navier-Stokes equations (1 step straitified Lie group)

$\boldsymbol{u} \in C([0,T), J^{3}(\mathbb{R}^3))$ $(J^3(\mathbb{R}^3) = L^3(\mathbb{R}^3)^3,\ \mathrm{div}\,\boldsymbol{w} = 0)$

- $V_1 = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$, $[V_1, V_1] = 0$
    - 解空間の次元は, $\mathrm{dim V_1}$ に依存.
    - 解空間の可積分性は $\mathrm{dim V_1}$, $\mathrm{dim V_2}$, 群の階層に依存.

結言

まとめ

Straitified Lie group の視点から Navier-Stokes 型方程式の初期値問題を考察すると,

- 方程式は左不変ベクトル場, 初期値や解空間は右不変ベクトル場で構成されている.
- 解空間の次元は, 第 1 層 V_1 の次元に依存する.
- 解空間の可積分性は, 全ての層 V_i の次元とその階層に依存する.

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