３重臨界点をもつ反応拡散系に現れる振動パターン

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偏微分方程式と現象：
PDEs and phenomena in Miyazaki 2011

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Motivation:

- Let us consider the variations of bifurcations from the stationary solutions .
- Are there possibilities that we can compute the bifurcation equation from non-uniform steady state ?
- One of the answer is double degeneracy of n - $n+1$ modal interaction induced by Turing instability.
- In this talk, we would like to discuss the other possibilities, that is, triple degenerate case.

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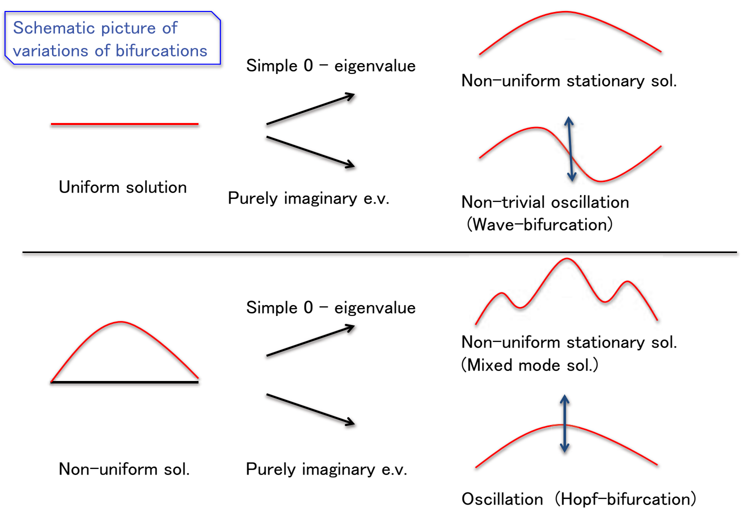

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Linear stability and triple degeneracy:

- The following RD system is easy to understand the mechanism for the triple degeneracy:

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u,v), & x \in (0,L), t \geq 0, \\ \tau_1 v_t = D_2 v_{xx} + u - v + G(u,v), & x \in (0,L), t \geq 0 \\ \tau_2 w_t = D_3 w_{xx} + u - w, & x \in (0,L), t \geq 0 \\ u_x = v_x = w_x = 0, & \text{at } x = 0, L, \end{cases} \quad (1)$$

We suppose that

- the time constant τ_j are very small;
- the diffusion constant D_3 is very large.

Setting $\tau_1 = \tau_2 = 0$. Let us consider only the linear terms in (1): Using the Fourier transformation, we have:

$$\begin{cases} \frac{d\hat{u}_k}{dt} = -Dk^2\hat{u}_k + a\hat{u}_k + b\hat{v}_k + s\hat{w}_k, \\ 0 = -D_2k^2\hat{v}_k + \hat{u}_k - \hat{v}_k, \\ 0 = -D_3k^2\hat{w}_k + \hat{u}_k - \hat{w}_k. \end{cases}$$

The second and third equations of above can be solved as

$$\hat{v}_k = \frac{\hat{u}_k}{1 + D_2k^2}, \quad \hat{w}_k = \frac{\hat{u}_k}{1 + D_3k^2}$$

Therefore, we obtain

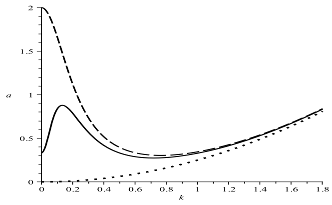
$$\frac{d\hat{u}_k}{dt} = \lambda_k \hat{u}_k,$$

where

$$\lambda_k = a - D_1k^2 + \frac{b}{1 + D_2k^2} + \frac{s}{1 + D_2k^2}.$$

Now we define the neutral stability curve by

$$C = \{(k, a)\,;\, \lambda_k = 0\}.$$



Constants are $D_1 = 0.25,\ D_2 = 20,\ D_3 = 100$.

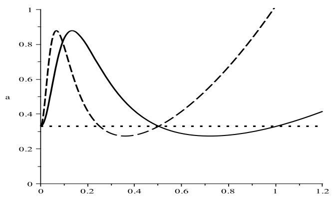
Dotted line: $b = s = 0$;

dashed line: $s = 0, b = -2$;

Solid line: $b = -2, s \approx 1.67$.

Since we are considering the Neumann boundary conditions, the wave number k should be an integer multiple of the fundamental wave number $k_0 = \pi/L$: $k = mk_0$

$$C_m = \{(k_0, a)\,;\ \lambda_{mk_0} = 0\}.$$



Constants are $D_1 = 0.25,\ D_2 = 20,\ D_3 = 100,\ b = 2,\ s \approx 1.67$.

Dotted line: C_0;

Solid line: C_1;

Dashed line: C_2 ;

The triple degeneracy of 0:1:2 interaction occur at

$$(k_0, a) \approx (0.50, 0.33).$$

2-component RD system with positive global feedback

Let us consider the following RD system:

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u,v), & x \in (0,L),\ t > 0, \\ v_t = D_2 v_{xx} + cu + dv + G(u,v), & x \in (0,L),\ t > 0, \\ \tau w_t = D_3 w_{xx} + u - w, & x \in (0,L),\ t > 0, \\ u_x = v_x = w_x = 0, & x = 0, L. \end{cases} \quad (2)$$

- Above system consists of two component activator-inhibitor type reaction-diffusion equations and one scalar equation which has the feedback effect to the first component.

It can be reduced in simpler system of reaction-diffusion equations as follows; putting $\tau = 0$, we have

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u,v), & x \in (0,L),\ t > 0, \\ v_t = D_2 v_{xx} + cu + dv + G(u,v), & x \in (0,L),\ t > 0, \\ 0 = D_3 w_{xx} + u - w, & x \in (0,L),\ t > 0, \\ u_x = v_x = w_x = 0, & x = 0, L. \end{cases} \quad (3)$$

If a solution $(u(t,x), v(t,x), w(x))$ of (3) can be represented in Fourier series, the third equation of (3) yields

$$0 = -(\pi/L)^2 m^2 D_3 w_m + u_m - w_m, \quad m \in \{0\} \cup \mathbb{N}.$$

It can be solved as

$$w_m = \frac{1}{1 + (\pi/L)^2 m^2 D_3} u_m, \quad m \in \{0\} \cup \mathbb{N}.$$

It holds that $w_0 = u_0$, and taking $D_3 \to \infty$, then

$$w_m \to 0, \quad m \in \mathbb{N}.$$

Combining them, we obtain

$$\begin{aligned} w(x) &= w_0 + \sum_{m \in \mathbb{N}} w_m \cos(m\pi x/L) \\ &= u_0 + \sum_{m \in \mathbb{N}} (1 + (\pi/L)^2 m^2 D_3)^{-1} u_m \cos(m\pi x/L) \\ &\to u_0 \qquad (\text{ as } D_3 \to \infty) \\ &= \frac{1}{L} \int_0^L u(x)\ dx. \end{aligned}$$

Substituting it into (2), we obtain the reaction-diffusion system with positive global feed back as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \dfrac{\partial^2 u}{\partial x^2} + au + bv + F(u,v) + \dfrac{s}{L}\displaystyle\int_0^L u(t,x)\ dx, \ \ x \in (0,L), t > 0, \\ \dfrac{\partial v}{\partial t} = D_2 \dfrac{\partial^2 v}{\partial x^2} + cu + dv + G(u,v), \ \ x \in (0,L), t > 0, \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0 \text{ at } x = 0, L. \end{cases} \tag{4}$$

Assumptions:

We consider the system (4) in a function space

$$X := \left\{ (u,v) \in [H(\Omega)]^2; u_x = v_x = 0 \text{ at } x = 0, L \right\},$$

where $\Omega = (0,L) \subset \mathbb{R}$.

- (A1) The functions (higher order terms) F and G are sufficiently smooth and $F(0,0) = G(0,0) = 0$;
- (A2) $F(u,v) \equiv -F(-u,-v)$ and $G(u,v) \equiv -G(-u,-v)$ hold.

- (A3) The following hold:

$$a, c > 0,\ b, d < 0,$$
$$a + d < 0,\ \Delta := ad - bc > 0;$$

- (A4) $\dfrac{bc}{d} + d < 0$ holds;

Then functions F and G can be presented by the taylor series around the origin:

$$\begin{aligned} F(u,v) &= \sum_{j+k=3} f_{jk} u^j v^k + o(\|u+v\|^3), \\ G(u,v) &= \sum_{j+k=3} g_{jk} u^j v^k + o(\|u+v\|^3). \end{aligned}$$

Dynamical system on Fourier space:

If $(u(t,x), v(t,x))$ is a solution of (4), then we can extend it for $x \in [0, 2L]$ as follows:

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & x \in [0,L], \\ u(t,2L-x) & x \in [L,2L], \end{cases} \quad \tilde{v}(t,x) = \begin{cases} v(t,x) & x \in [0,L], \\ v(t,2L-x) & x \in [L,2L]. \end{cases}$$

Then, $(\tilde{u}(t,x), \tilde{v}(t,x))$ is a solution of

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + F(u,v) + \frac{s}{2L}\int_0^{2L} u(t,x)\,dx, \ \ x \in \Omega_p, t > 0, \\ v_t = D_2 v_{xx} + cu + dv + G(u,v), \ \ x \in \Omega_p, t > 0, \\ u(t,x) = u(t,x+2L), u_x(t,x) = u_x(t,x+2L) \ \ t > 0, \\ v(t,x) = v(t,x+2L), v_x(t,x) = v_x(t,x+2L) \ \ , t > 0. \end{cases} \tag{5}$$

Here, Ω_p denotes the interval $(0, 2L)$.

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Thus, we consider the system (5) in a function space

$X_p := \{(u,v) \in [H^2_{per}(\Omega_p)]^2; (u(x), v(x)) = (u(2L-x), v(2L-x))\}$

instead of (4). Substituting

$$(u(t,x), v(t,x)) = \sum_{m \in \mathbb{Z}} (u_m(t), v_m(t)) e^{imk_0 x}, \tag{6}$$

in to (5), We have

$$\frac{d}{dt}\begin{pmatrix} u_m \\ v_m \end{pmatrix} = M_m \begin{pmatrix} u_m \\ v_m \end{pmatrix} + \begin{pmatrix} f_m \\ g_m \end{pmatrix}, m \geq 0. \tag{7}$$

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Here,

$$M_0 = \begin{pmatrix} a+s & b \\ c & d \end{pmatrix},$$

$$M_m = \begin{pmatrix} a - D_1 m^2 k_0^2 & b \\ c & d - D_2 m^2 k_0^2 \end{pmatrix}, \ (m \neq 0),$$

$$f_m = \sum_{m_1+m_2+m_3=m} (f_{30} u_{m_1} u_{m_2} u_{m_3} + f_{21} u_{m_1} u_{m_2} v_{m_3} + f_{12} u_{m_1} v_{m_2} v_{m_3} + f_{03} v_{m_1} v_{m_2} v_{m_3}),$$

$$g_m = \sum_{m_1+m_2+m_3=m} (g_{30} u_{m_1} u_{m_2} u_{m_3} + g_{21} u_{m_1} u_{m_2} v_{m_3} + g_{12} u_{m_1} v_{m_2} v_{m_3} + g_{03} v_{m_1} v_{m_2} v_{m_3}).$$

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0:1:2 triple degeneracy in (4)

Triple degeneracy is obtained by solving

$$\det M_0 = \det M_j = \det M_\ell = 0,\ j \neq \ell,\ j,\ell \in \mathbb{N}.$$

Then, we have

Lemma 1 *For given two positive integers $j,\ell,(\,j \neq \ell)$ and constants $D_1, a, b, c, d,$ there exit positive constants $k_0^{j,\ell}, D_2^{j,\ell}$ and s^* such that if $(k_0, D_2, s) = (k_0^{j,\ell}, D_2^{j,\ell}, s^*)$ then the linearized eigenvalue problem of (7) about a trivial solution has strictly three zero eigenvalues.*

We can compute $D^{j,\ell}, k_0^{j,\ell}$ and s^* directly:

$$\begin{aligned} k_0^{j,\ell} &= \left[\frac{1}{2dD_1 j^2\ell^2}\left\{\Delta(j^2+\ell^2) - \sqrt{\Delta^2(j^2+\ell^2)^2 - 4ad\Delta j^2\ell^2}\right\}\right]^{1/2}, \\ D_2^{j,\ell} &= \frac{\{dD_1 j^2 (k_0^{j,\ell})^2 - \Delta\}}{j^2(k_0^{j,\ell})^2\{D_1 j^2 (k_0^{j,\ell})^2 - a\}}, \\ s^* &= -\Delta/d = -(ad-bc)/d. \end{aligned}$$

- This yields a triple degeneracy of 0:1:2 interaction by choosing $j = 1$ and $\ell = 2$.

Center manifold reduction

Let us derive the normal form on the center manifolds around the triply degenerate point $(k_0^{1,2}, D^{1,2}, s^*)$ of (4).

Diagonalization:

We diagonalize the equations in (7) for $m = 0,1,2$. Set $(k_0, D_2, s) = (k_0^{1,2}, D_2^{1,2}, s^*)$. Then changing variables ${}^t(u_m, v_m) = T_m\, {}^t(\tilde{u}_m, \tilde{v}_m), (m = 0,1,2)$ by the matrix

$$T_0 = \begin{pmatrix} -d & bc/d \\ c & c \end{pmatrix},\ T_m = \begin{pmatrix} -d + D_2 m^2 k_0^2 & a - D_1 m^2 k_0^2 \\ c & c \end{pmatrix}, m = 1,2,$$

we have

$$\begin{pmatrix} \dot{\tilde{u}}_m \\ \dot{\tilde{v}}_m \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_m^- \end{pmatrix}\begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} + T_m^{-1}\begin{pmatrix} \tilde{f}_m \\ \tilde{g}_m \end{pmatrix}, m = 0,1,2.$$

Here,

$$\lambda_0^- := d + bc/d,$$

$$\lambda_m^- := (a+d) - m^2(D_1 + D_2^{1,2})(k_0^{1,2})^2,$$

$$\tilde{f}_m := f_m|_{{}^t(u_{m_j}, v_{m_j}) = T_m {}^t(\tilde{u}_{m_j}, \tilde{v}_{m_j})},$$

$$\tilde{g}_m := g_m|_{{}^t(u_{m_j}, v_{m_j}) = T_m {}^t(\tilde{u}_{m_j}, \tilde{v}_{m_j})}.$$

Then, the following holds:

Theorem 1 *The dynamics of (7) on the center manifold (of (7)) can be approximated by the following system :*

$$\begin{cases} \dot{z}_0 = (\mu_0 + a_1 z_0^2 + a_2 z_1^2 + a_3 z_2^2) z_0 + a_4 z_1^2 z_2, \\ \dot{z}_1 = (\mu_1 + b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 z_0 z_1 z_2, \\ \dot{z}_2 = (\mu_2 + c_1 z_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 z_0 z_1^2. \end{cases} \tag{8}$$

Here, z_j denote $\tilde{u}_j$ $(j = 0, 1, 2)$, and $o(3)$ denotes $o(|(z_0, z_1, z_2)|^3)$. In addition, the coefficients μ_j, a_j, b_j, c_j are dependent on the coefficients and parameters appearing in (7).

Hopf-instability around the 1-mode equilibriums:

Linearized matrix of $\pm e_1 := (0, \pm\sqrt{-\mu_1/b_2}, 0)$ is given by

$$M_{\pm e_1} := \begin{pmatrix} \mu_0 - \dfrac{a_2}{b_2}\mu_1 & 0 & \dfrac{-a_4}{b_2}\mu_1 \\ 0 & -2\mu_1 & 0 \\ -\dfrac{c_4}{b_2}\mu_1 & 0 & \mu_2 - \dfrac{c_2}{b_2}\mu_1 \end{pmatrix}.$$

Put

$$\tilde{M}_{\pm e1} := \begin{pmatrix} \mu_0 - \dfrac{a_2}{b_2}\mu_1 & \dfrac{-a_4}{b_2}\mu_1 \\ -\dfrac{c_4}{b_2}\mu_1 & \mu_2 - \dfrac{c_2}{b_2}\mu_1 \end{pmatrix}.$$

Put $z_{p1} = \sqrt{-\mu_1/b_2}$. Then if

$$\text{tr}\,\tilde{M}_{\pm e_1} = 0 \text{ and } \det \tilde{M}_{\pm e_1} > 0,$$

or more precisely, if

$$(\mu_0 + a_2 z_{p1})^2 + a_4 c_4 z_{p1}^4 < 0$$

then the matrix $M_{\pm e_1}$ has a pair of purely imaginary eigenvalues at

$$\mu_2 = -\mu_0 - (a_2 + c_2) z_{p1}^2.$$

We can conclude that; (i) $a_4 c_4 < 0$ is necessary; (ii) If $\mu_0 = a_2 \mu_1 / b_2$ then $\det \tilde{M}_{\pm e1}$ attain a minimum $a_4 c_4$.

We can compute the normal form for the Hopf-bifurcation as follows:

Lemma 2 *If $\mu_1 b_2 < 0$ and $\tilde{M}_{\pm e_1} > 0$ then, the norma form for the Hopf-bifurcation around the equilibrium $(0, \sqrt{-\mu_1/b_2}, 0)$ is given by the following:*

$$\frac{dz}{d\tau} = (\beta + i) z + \varsigma |z|^2 z + o(|z|^3), \tag{9}$$

where z and τ are a new complex coordinate and the new time, respectively, and β is a new parameter, and moreover,ς is dependent on the coefficient in (8), and computable.

$$\begin{aligned}\varsigma \;=\; \text{sign}\Big[& \frac{1}{2c_4 z_{p1}^2}(2c_4 z_{p1}^2 c_1 \omega^2 + 6c_4 z_{p1}^2 c_1 \nu^2 + 12 c_4 z_{p1}^3 c_2 \gamma_{20} \\ & + 4c_4 z_{p1}^3 c_2 \gamma_{02} + 6c_4^3 z_{p1}^6 c_3 + 12 c_4 z_{p1} \nu \gamma_{20} + 4 c_4 z_{p1} \nu \gamma_{02} + 4 c_4 z_{p1} \omega \gamma_{11}) \\ & + \frac{1}{z_{p1}\omega}(3 z_{p1} a_1 \omega^3 + 3 z_{p1} \omega a_1 \nu^2 + 2\omega a_2 z_{p1}^2 \gamma_{20} + 6\omega a_2 z_{p1}^2 \gamma_{02} + \omega a_3 c_4^2 z_{p1}^5 \\ & + 2\nu a_2 z_{p1}^2 \gamma_{11} + 2 a_4 z_{p1}^4 c_4 \gamma_{11} - 2 z_{p1} c_1 \omega \nu^2 \\ & - 2\nu z_{p1}^2 c_2 \gamma_{11} - 2\nu\omega\gamma_{20} - 6\nu\omega\gamma_{02} - 2\nu^2 \gamma_{11} \Big],\end{aligned}$$

$$\gamma_{20} = \frac{-1}{4b_2 z_{p1}(\omega^2 + b_2^2 z_{p1}^4)}(-2b_2 z_{p1}^2 b_1 \omega^2 \nu - b_2 z_{p1}^4 b_4 \omega^2 c_4 + \omega^4 b_1$$
$$+\omega^2 b_3 c_4^2 z_{p1}^4 + \omega^2 c_4 \nu b_4 z_{p1}^2 + \omega^2 b_1 \nu^2 + 2b_2^2 z_{p1}^4 b_1 \nu^2$$
$$+2b_2^2 z_{p1}^8 b_3 c_4^2 + 2b_2^2 z_{p1}^6 c_4 \nu b_4),$$

$$\gamma_{11} = \frac{-\omega z_{p1}}{2(\omega^2 + b_2^2 z_{p1}^4)}(b_3 c_4^2 z_{p1}^4 + c_4 \nu b_4 z_{p1}^2 + 2b_2 z_{p1}^2 b_1 \nu$$
$$+b_2 z_{p1}^4 b_4 c_4 - \omega^2 b_1 + b_1 \nu^2),$$

$$\gamma_{02} = \frac{-\omega^2}{4b_2 z_{p1}(\omega^2 + b_2^2 z_{p1}^4)}(b_3 c_4^2 z_{p1}^4 + c_4 \nu b_4 z_{p1}^2 + 2b_2 z_{p1}^2 b_1 \nu + b_2 z_{p1}^4 b_4 c_4$$
$$+\omega^2 b_1 + b_1 \nu^2 + 2b_2^2 z_{p1}^4 b_1)$$

($z_{p1} = \sqrt{-\mu_1/b_2}$, $\nu = \mu_0 + a_2 z_{p1}^2, \omega = \sqrt{-\nu^2 - a_4 c_4 z_{p1}}$).

In the above case, the coefficient ς is dependent on the parameter μ_0 even though μ_1 is fixed so that 1-mode stationary solutions $\pm e_1$ exist.

Here we consider the simple case of

$$(\mu_0, \mu_2) = \left(\frac{a_2}{b_2}\mu_1, \frac{c_2}{b_2}\mu_1\right).$$

In this case,

$$M_{\pm e_1} := \begin{pmatrix} 0 & 0 & a_4 z_{p1}^2 \\ 0 & -2\mu_1 & 0 \\ c_4 z_{p1}^2 & 0 & 0 \end{pmatrix}.$$

If $a_4 c_4 < 0$ then the matrix has a pair of purely imagenary eigenvalues. And in this case, the coefficient ς is independent of the parameters as follows:

$$\varsigma = \text{sign} \left[a_4^2(3P_1 + P_2) - a_4 c_4 (P_3 + 3P_4)\right]. \qquad (10)$$

Here,

$$P_1 = \frac{1}{2b_2(a_4 c_4 - b_2^2)}(2a_1 b_2 c_4 a_4 - 2a_1 b_2^3 - a_2 c_4 a_4 b_1$$
$$+a_2 c_4 b_4 b_2 + a_2 c_4^2 b_3 + 2a_2 b_1 b_2^2),$$

$$P_2 = \frac{1}{2b_2(a_4 c_4 - b_2^2)}(-2c_1 b_2^3 + 2c_1 b_2 c_4 a_4 - c_2 c_4 a_4 b_1 + c_2 c_4 b_4 b_2$$
$$+c_2 c_4^2 b_3 + 2c_2 b_1 b_2^2 + 2b_2 c_4 a_4 b_1 + 2b_2^2 c_4 b_4 + 2c_4^2 b_3),$$

$$P_3 = -\frac{1}{2b_2(a_4 c_4 - b_2^2)}(a_2 a_4^2 b_1 + a_2 a_4 b_4 b_2 - a_2 a_4 c_4 b_3 + 2a_2 b_3 b_2^2$$
$$+2a_3 b_2^3 - 2a_3 b_2 c_4 a_4 - 2b_2 a_4^2 b_1 - 2b_2^2 a_4 b_4 - 2b_2 a_4 c_4 b_3),$$

$$P_4 = -\frac{1}{2b_2(a_4 c_4 - b_2^2)}(c_2 a_4^2 b_1 + c_2 a_4 b_4 b_2 - c_2 a_4 c_4 b_3$$
$$+2c_2 b_3 b_2^2 + 2c_3 b_2^3 - 2c_3 b_2 c_4 a_4).$$

(We use this formulation to compute the ς in the latter of this talk).

Remark

If $a_2 > 0$, then the time-periodic solutions around 1-mode stationary solutions exist even though the eigenvalues corresponding to the spatially uniform eigenfunction (0-mode) are negative.

A case study:

Let us study the case where

$$\begin{aligned} &D_1 = 1/4, a = 1, b = -10, c = 2, d = -5, \\ &F(u,v) = -u^3, G(u,v) = -0.9u^3. \end{aligned} \tag{11}$$

Then we have $s^* = -3$ and

$$(k_0^{1,2}, D_2^{1,2}) \approx (0.87, 25.88).$$

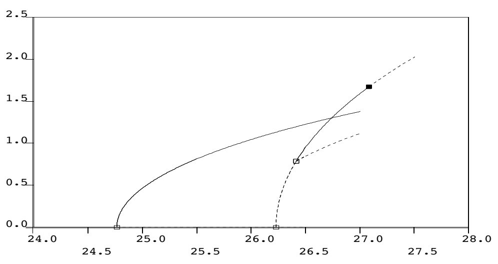
The coefficients of (8) are

$$\begin{aligned} &a_1 \approx 100.00, \quad a_2 \approx 14644.17, \quad a_3 \approx 1.69 \times 10^5, \quad a_4 \approx 2.45 \times 10^5, \\ &b_1 \approx -49.29, \quad b_2 \approx -1203.01, \quad b_3 \approx -27695.10, \quad b_4 \approx -1652.32, \\ &c_1 \approx -67.14, \quad c_2 \approx -3277.22, \quad c_3 \approx -18861.66, \quad c_4 \approx -97.761. \end{aligned} \tag{12}$$

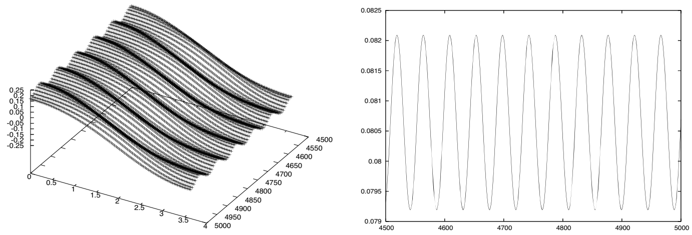
It follows that $a_4 c_4 < 0$ and $\varsigma = -1$. Therefore, time-periodic solutions around 1-mode stationary solutions exist, and moreover, they are locally asymptotically stable in this case.

Numerical results to RD system:

We show the numerical results which agree with the normal form analysis.



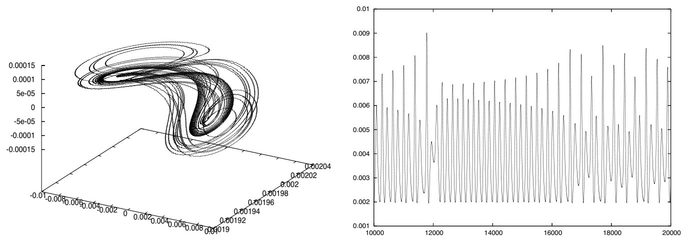
Bifurcation diagram of RD system (4) in (11). (Vertical axis: L^2 norm of (u,v), horizontal axis: D_2). The Hopf-bifurcation point (which is symbolized a black square) appears on a branch of 1-mode stationary solution.



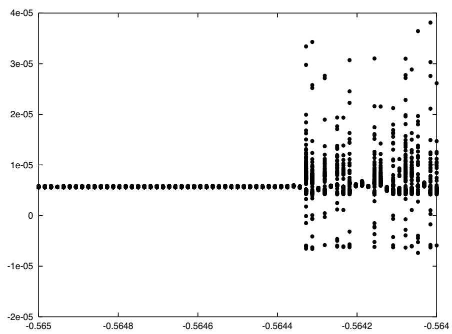
Numerical results RD system (4) with $D_2 = 27.13$ in the case (11).
(Left:) Graph of $u(t, x)$, $(t \in [4500, 5000], x \in [0, L])$.
(Right:) The graph of $\|(u, v)\|_{L^2}(t)$, $t \in [4500, 5000]$

Numerical studies to chaotic behavior in (8)

We show the numerical results:
$\mu_0 = -0.559489, \quad \mu_1 = 0.0052, \quad \mu_2 = -0.002.$



Numerical results to (8) in the case of (12)
(Left:) Orbits of solutions.
(Right:) Euclidean norm vs. time t.



Bifurcation diagram of Poincaré map of (8) at $z_0 = z_{0_*}$
(z_{0_*} is a coordinate of the equilibriums : $(z_0(t), z_1(t), z_2(t)) = (z_{0_*}, z_{1_*}, z_{2_*})$).
The parameters: $\mu_1 = 0.0052$, $\mu_2 = -0.002$. The vartical and horizontal axes correspond to z_3 and μ_0.
The periodic orbit disappear at $\mu_0 \approx 0.5644$ and it changed to the chaotic attractor.

Equilibriums of (8) in the case (12) and their linearized eigenvalues.

Equilibriums	Eigenvalues		
$(0, 0, 0)$	-0.564489	0.0052	-0.0020
$(\pm 0.0748, 0, 0)$	-0.3776	-0.2706	1.1190
$(0, \pm 0.0021, 0)$	0.0691	-0.0109	-0.0104
$\pm(0.0042, 0.00200, -0.0001)$	-0.0140	0.0071 $\pm 0.0324i$	
$\pm(0.0042, -0.00200, -0.0001)$	-0.0140	0.0071 $\pm 0.0324i$	

- All equilibriums are unstable. The solution can not converge to the equilibriums.

Is the complex dynamics in (8) cahos ?

We compute the Lyapunov characteristic exponents. The following properties are convenient to compute the Lyapunov exponents.

Scale-invariance of the normal form:

- The system (8) is invariant under the scaling:

$$\tilde{z}_j = \eta z_j, \quad \tilde{\mu}_j = \eta^2 \mu_j, \quad \tilde{t} = \eta^2 t, \quad {}^{\forall}\eta \in \mathbb{R}.$$

Using this invariance, we can magnify the amplitude of numerical solutions.

Remark:

(I) If higher order terms are $O(|(z_0, z_1, z_2)|^5)$, the scaled system is

$$\tilde{z}_j' = \tilde{\mu}_j \tilde{z}_j + f_j(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2; \tilde{\mu}_j) + \eta^2 O(|(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2)|^5),$$

where f_j are cubic terms in (8). Taking $\eta \to 0$, the higher order terms are banished asymptotically.

(II) If there is a (large amplitude) solution $z_j(t; \mu_j)$ of (8) in $t \in (0, T]$, there exist a similar, and small amplitude solution of (8) :

$$\tilde{z}_j(\tilde{t}; \tilde{\mu}_j) = \eta z_j(\eta^2 t; \eta^2 \mu_j), \quad t \in (0, T]$$

by taking η small.

How to compute the Lyapunov exponents:

We use the algorithm shown in
I. Shimada and T. Nagashima, Prog. Theor. Phys. 61 (1979) 1605 – 1616.

Let

$$\dot{\mathbf{z}} = F(\mathbf{z}) \tag{13}$$

be a system of differential equation in $\mathbb{R}^3$, and let $\{\mathbf{e}_j\}$, $j = 1, 2, 3$ be a set of basis of tangent space at $\mathbf{z} = \mathbf{z}_0 := \mathbf{z}(0)$. Consider the variational equation around the flow $\mathbf{z}(t; \mathbf{z}_0)$:

$$\dot{\mathbf{y}}(t) = DF(\mathbf{z}(t; \mathbf{z}_0))\mathbf{y}(t). \tag{14}$$

Then, the solution of it can be written as $\mathbf{y}(t) = U^t\mathbf{y}(0)$, where U^t is the fundamental matrix.

We define

$$\lambda(e^1, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1|/|\mathbf{e}_1|\right),$$

$$\lambda(e^2, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1 \times U^t\mathbf{e}_2|/|\mathbf{e}_1 \times \mathbf{e}_2|\right),$$

$$\lambda(e^3, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1 \cdot (U^t\mathbf{e}_2 \times U^t\mathbf{e}_3)|/|\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)|\right),$$

where e^j are j-dimensional space defined by $e^j = span\{\mathbf{e}_1, \dots, \mathbf{e}_j\} \subset \mathbb{R}^3$, and $\circ \cdot \circ$ and $\circ \times \circ$ denote inner product and exterior product.

Then, the Lyapunov exponents λ_j, $(\lambda_1 > \lambda_2 > \lambda_3)$ satisfy the following:

$$\lambda(e^1, \mathbf{z}_0) = \text{one of values in } \{\lambda_1, \lambda_2, \lambda_3\},$$

$$\lambda(e^2, \mathbf{z}_0) = \text{one of values in } \{(\lambda_1 + \lambda_2), (\lambda_2 + \lambda_3), (\lambda_3 + \lambda_1\},$$

$$\lambda(e^3, \mathbf{z}_0) = \lambda_1 + \lambda_2 + \lambda_3.$$

Classification of the attractors:

Attractor	sign of λ_1	sign of λ_2	sign of λ_3
Fixed point	–	–	–
Limit cycle	0	–	–
Torus	0	0	0
Strange attractor	+	0	–

Equations:

$$\begin{cases} \dot{z}_0 = (\mu_0 + a_1 z_0^2 + a_2 z_1^2 + a_3 z_2^2) z_0 + a_4 z_1^2 z_2, \\ \dot{z}_1 = (\mu_1 + b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 z_0 z_1 z_2, \\ \dot{z}_2 = (\mu_2 + c_1 z_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 z_0 z_1^2. \end{cases}$$

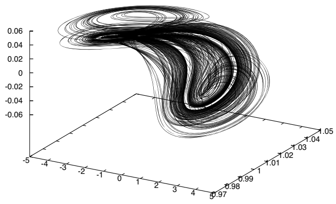
Constants

$\mu_0 = -0.559489$, $\mu_1 = 0.0052$, $\mu_2 = -0.002$,
$a_1 \approx 100.00$, $a_2 \approx 14644.17$, $a_3 \approx 1.69 \times 10^5$, $a_4 \approx 2.45 \times 10^5$,
$b_1 \approx -49.29$, $b_2 \approx -1203.01$, $b_3 \approx -27695.10$, $b_4 \approx -1652.32$,
$c_1 \approx -67.14$, $c_2 \approx -3277.22$, $c_3 \approx -18861.66$, $c_4 \approx -97.761$.

- Scheme: Runge-Kutta-Fehlberg method (errors of the solution are smaller than 10^{-6}) ;
- Time difference: 10^{-2} ;
- Scaling parameters: $\eta = 2^2$;
- Normalization: The bases normalized by Gram-Schmidt orthonormalization in each step;
- Time : $0 < t < T$

The Lyapunov characteristic exponents for each T:

T	λ_1	λ_2	λ_3
5×10^3	0.014125	0.000286	−0.136404
10^4	0.021547	0.000112	−0.136693
5×10^4	0.025032	−0.000014	−0.137087
10^5	0.023560	−0.000000	−0.137040
5×10^5	0.023984	−0.000028	−0.137153
10^6	0.023906	−0.000024	−0.137146
5×10^6	0.023768	−0.000025	−0.137136



Attractor with the scaling.

Then we estimate

$$\lambda_1 \approx 0.0240, \quad \lambda_2 \approx 0.0000, \quad \lambda_3 \approx -0.1371.$$

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Lyapunov dimension of the attractor as follows
Definition:
(for instance, see J. Kaplan and J. York: Lecture notes in mathematics, vol. 730):
Let j be a integer satisfying

$$\sum_{\ell=1}^{j} \lambda_\ell > 0 \text{ and } \sum_{\ell=1}^{j+1} \lambda_\ell < 0,$$

then the Lyapunov dimension d_f is defined by

$$d_f = j + \frac{\sum_{\ell=1}^{j} \lambda_\ell}{\lambda_{j+1}}.$$

In our case, we obtain

$$d_f \approx 2.175.$$

(cf. : Lorenz-attractor : 2.07, Rossler-attractor: 2.01;
[see A. Wolf, et al., Physica 16 D (1985), 285 – 317])

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- The normal form (8) yields chaotic dynamics by a suitable choice of parameters and coefficients,
- The reaction-diffusion system has a chaotic solution on the center manifold in a suitable settings of nonlinearity and coefficients (see next slide).

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Chaotic behavior in RD system with positive feed back

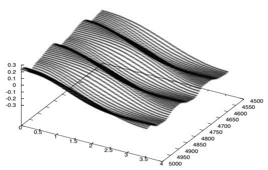
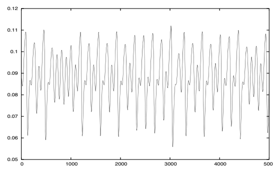
Our RD system is

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \dfrac{\partial^2 u}{\partial x^2} + F(u,v) + \dfrac{s}{L}\displaystyle\int_0^L u(t,x)\ dx, \ \ x \in (0,L), t > 0, \\ \dfrac{\partial v}{\partial t} = D_2 \dfrac{\partial^2 v}{\partial x^2} + G(u,v), \ \ x \in (0,L), t > 0, \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0 \text{ at } x = 0, L. \end{cases}$$

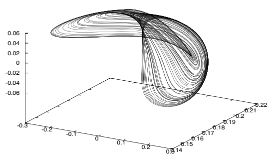
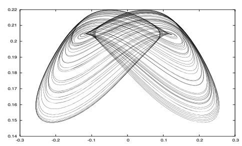
Constants:

$$D_1 = 1/4, a = 1, b = -10, c = 2, d = -5,$$
$$D_2 = 26.5, k_0 = 0.874919 = (\pi/L), s = 2.978084$$
$$F(u,v) = -u^3, G(u,v) = -0.9u^3.$$

(Critical point is $(k_0, D_2, s) \approx (0.87, 25.88, 3)$)

(Left:) Graph of $u(t,x)$, (Right:) Graph of $\|u\|_{L^2}(t)$

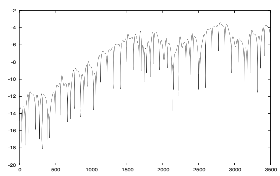



(Left:) Projection on Fourier sapce (u_0, u_1, u_2),
(Right:)Projection on Fourier sapce (u_0, u_1).

Numerical experiment: check the sensitivity to initial conditions

Put $(u_0^{(2)}(x), v_0^{(2)}(x)) := (u_0^{(1)}(x), v_0^{(1)}(x)) + (10^{-6}, 0)$.
$(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ are solutions of (4) satisfying

$$(u^{(j)}(0,x), v^{(j)}(0,x)) = (u_0^{(j)}(x), v_0^{(j)}(x)),\ j = 1,2.$$



The graph of $\log|\ \|u^{(1)}\|_{L^2} - \|u^{(2)}\|_{L^2}\ |(t)$.

Conclusions and remarks

- We compute normal form around a triply degenerate point, and show a typical case in which (4) has time-periodic orbits around one mode stationary solutions, and chaotic attractors.
- The mechanism of the chaotic dynamics is not clear in this talk. It is going to be a future problem.
- If the Neumann boundary conditions are replaced with periodic b.c. , the dynamics around 0:1:2 degenerate point is given as a ODE system on $\mathbb{C}^2 \times \mathbb{R}$. There could be more complex spatiotemporal patterns (it is also going to be a future problem).