応用数学 II No.12 解答

1.

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\omega x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i\omega x}}{i\omega}\right]_a^b = \frac{i(e^{-i\omega b} - e^{-i\omega a})}{\omega\sqrt{2\pi}}\end{aligned}$$

2. (1) $|e^{i\omega x}| = 1, \lim_{x\to\infty} xe^{-x} = 0$ より、

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x\, e^{-(1+i\omega)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \left[-\frac{xe^{-(1+i\omega)x}}{1+i\omega}\right]_0^{\infty} + \frac{1}{1+i\omega} \int_0^{\infty} e^{-(1+i\omega)x}\, dx \right\} \\ &= \frac{1}{\sqrt{2\pi}}\, \frac{-1}{(1+i\omega)^2} [e^{-(1+i\omega)x}]_0^{\infty} = \frac{1}{\sqrt{2\pi}}\, \frac{1}{(1+i\omega)^2}.\end{aligned}$$

(2) $g(x) = h(x) = e^{-x} \quad (x > 0), \quad g(x) = h(x) = 0 \quad (x > 0)$ とする。

$p > 0$ そして $x - p > 0$ より $0 < p < x$.

$$(g*h)(x) = \int_0^x g(p)h(x-p)dp = \int_0^x e^{-p}\, e^{-(x-p)}\, dp = \int_0^x e^{-x}\, dp = xe^{-x} \quad (x > 0)$$

$x < 0$ のとき $(g*h)(x) = 0$.

$$\mathscr{F}(g(x)) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x}\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-(1+i\omega)x}}{1+i\omega}\right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega}.$$

$$\hat{f}(\omega) = \mathscr{F}(g*h)(x) = \sqrt{2\pi}\, \mathscr{F}(g(x))\mathscr{F}(h(x)) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+i\omega)^2}.$$

3.

$$\begin{aligned}\mathscr{F}(e^{iax} f(x)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(\omega-a)x}\, dx \\ &= \hat{f}(\omega - a).\end{aligned}$$