数学解析 II No. 7 解答

1. (1) $y' = \dfrac{x^2}{4} - \dfrac{1}{x^2}$ となる。

$$\int_1^2 \sqrt{1+y'^2}dx = \int_1^2 \sqrt{1+\Big(\frac{x^2}{4}-\frac{1}{x^2}\Big)^2}dx = \int_1^2 \Big(\frac{x^2}{4}+\frac{1}{x^2}\Big)dx = \Big[\frac{x^3}{12}-\frac{1}{x}\Big]_1^2 = \frac{13}{12}.$$

(2) $x' = 6t,\ y' = 3-3t^2$ となる。

$$\int_0^{\sqrt{3}} \sqrt{x'^2+y'^2}dt = \int_0^{\sqrt{3}} \sqrt{(6t)^2+(3-3t^2)^2}dt = 3\int_0^{\sqrt{3}}(1+t^2)dt = 3\Big[t+\frac{t^3}{3}\Big]_0^{\sqrt{3}} = 6\sqrt{3}.$$

(3) $x' = a(1-\cos\theta),\ y' = a\sin\theta$ となる。$\cos\theta = 1-2\sin^2\dfrac{\theta}{2}$ と 変数変換 $\theta = 2t$ を用いると、

$$\begin{aligned}\int_0^{2\pi} \sqrt{x'^2+y'^2}d\theta &= \int_0^{2\pi} a\sqrt{(1-\cos\theta)^2+\sin^2\theta}d\theta = \sqrt{2}a\int_0^{2\pi}\sqrt{1-\cos\theta}d\theta \\ &= 2a\int_0^{2\pi}\Big|\sin\frac{\theta}{2}\Big|d\theta = 4a\int_0^{\pi}|\sin t|dt = 8a\int_0^{\frac{\pi}{2}}\sin t dt = 8a\Big[-\cos t\Big]_0^{\frac{\pi}{2}} = 8a.\end{aligned}$$

予備問題：

(1) $y' = \dfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$ となる。

$$\int_0^a \sqrt{1+y'^2}\ dx = \int_0^a \sqrt{1+\Big(\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\Big)^2}\ dx = \int_0^a \frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}\ dx = \frac{a}{2}\Big[e^{\frac{x}{a}}-e^{-\frac{x}{a}}\Big]_0^a = \frac{a(e-e^{-1})}{2}.$$

(2) $x' = e^t(\cos t - \sin t),\ y' = e^t(\sin t + \cos t)$ となる。

$$\int_0^{\pi} \sqrt{x'^2+y'^2}\ dt = \int_0^{\pi} \sqrt{2}e^t\ dt = \sqrt{2}\left[e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi}-1)$$