CONVERGENCE OF A CRYSTALLINE ALGORITHM FOR THE MOTION OF A CLOSED CONVEX CURVE BY A POWER OF CURVATURE $V = K^\alpha$  

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Abstract. A finite difference scheme is constructed for the evolution of a plane curve driven by a power of curvature: $V = K^\alpha$, $\alpha > 0$, where $V$ and $K$ are the normal velocity and the curvature, respectively. Here we assume that the curve is closed, strictly convex, and immersed in the plane $\mathbb{R}^2$. This curve-evolution is discretized using a crystalline approximation which is a kind of finite difference scheme for the nonlinear evolution equation. We prove convergence of this scheme and show the rate of convergence. Moreover, some numerical examples are presented.

Key words. crystalline curvature, crystalline algorithm, motion by curvature, curve-shortening, blow-up, numerical solution, rate of convergence

AMS subject classifications. 34, 35, 53, 65C, 65D, 65G, 65M

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1. Introduction. Let $\Gamma_0$ be a smooth closed convex curve and immersed in the plane $\mathbb{R}^2$. We consider the moving boundary problem of finding a family of closed curves $\{\Gamma(t)\}_{t \geq 0}$ which satisfies

\begin{align}
\begin{cases}
V(\theta,t) &\equiv K(\theta,t)^\alpha, \quad \theta \in T_\eta, \quad t \geq 0, \\
\Gamma(0) &= \Gamma_0,
\end{cases}
\end{align}

where $V$ is the inward normal component of the velocity of $\Gamma(t)$ and $K$ is the curvature with the sign convention that the curvature of a circle is positive. Parameter $\theta$ is the angle between the exterior normal and a fixed axis, $\alpha$ is a positive real-valued parameter and $t$ is often called time. Parameter $\eta \geq 1$ is winding number of $\Gamma(t)$ and $T_\eta := \mathbb{R}/2\eta \pi \mathbb{Z}$. The functions $V$ and $K$ are $2\eta \pi$-periodic in $\theta$ since curve $\Gamma(t)$ is closed.

Problem (1.1) is a so-called curve-shortening equation (see (1.5)). In the special case when $\alpha = 1$, (1.1) is called motion by mean curvature and investigated in many publications (see, e.g., [3], [5], [8], and references therein). The motion by mean curvature first appeared in material science [16]. Equation (1.1) is derived as the modeling of the motion of the grain boundary in annealing metals. In the case when $\alpha = \frac{1}{3}$, (1.1) is known as a model equation for image processing [1]. Also it is known that (1.1) does not change form under area-preserving affine transformation [18].

It is often convenient to describe the evolution of everywhere positive curvature $K$ under Gauss-parameterization of the curve $\Gamma(t)$. The following problem is equivalent to (1.1).
Problem 1.1. Find a function $V = V(\theta, t)$ satisfying

\begin{align}
(1.2a) \quad & V_t = \alpha V \frac{\partial}{\partial \theta} (V_{\theta\theta} + V), \quad \theta \in T_\eta, \quad 0 \leq t < T, \\
(1.2b) \quad & V(\theta, 0) = \varphi(\theta) > 0, \quad \theta \in T_\eta, \\
(1.2c) \quad & V(0, t) = V(2\eta\pi, t), \quad V_\theta(0, t) = V_\theta(2\eta\pi, t), \quad 0 \leq t < T, \\
(1.2d) \quad & \int_0^{2\eta\pi} \cos \frac{\xi}{\varphi(\xi)} d\xi = \int_0^{2\eta\pi} \sin \frac{\xi}{\varphi(\xi)} d\xi = 0,
\end{align}

where $\alpha > 0$ and $\eta \geq 1$ are real and natural numbers, respectively, and $V_t = \frac{\partial V}{\partial t}$, $V_{\theta\theta} = (V_\theta)_\theta$, $V_\theta = \frac{\partial V}{\partial \theta}$.

Equations (1.2b), (1.2c), and (1.2d) are the initial, the periodic boundary, and the closedness condition, respectively.

For a convex evolving curve with curvature $K = K(\theta, t)$ and inward normal $V = V(\theta, t)$ the following relation holds (see, e.g., (2.20) in [9]):

$$K_t = K^2 (V_{\theta\theta} + V).$$

Substituting (1.1) into the above equation, we obtain (1.2a). Conversely, for a given solution $V(\theta, t)$ of Problem 1.1, we can construct a family of curves $\{\Gamma(t)\}_{t \geq 0}$ which obeys the evolution law (1.1). In fact, the curve $\Gamma(t) = \{x \in \mathbb{R}^2 | x = X(\theta, t), \ \theta \in T_\eta, \ t > 0\}$ with $\Gamma_0 = \{x \in \mathbb{R}^2 | x = X_0(\theta), \ \theta \in T_\eta\}$ satisfies (1.1), where $\{X(\theta, t) | \theta \in T_\eta, \ t > 0, \ X(\theta, 0) = X_0(\theta)\}$ is a solution of the following:

\begin{align}
(1.3) \quad & X_t(\theta, t) = V(\theta, t)N(\theta), \quad \theta \in T_\eta, \quad t > 0, \\
(1.4) \quad & X_0(\theta) = X_0(0) + \int_0^\theta \frac{T(\xi)}{\varphi(\xi)} d\xi, \quad \theta \in T_\eta,
\end{align}

Here $N(\theta) = -(\cos \theta, \sin \theta)$ (respectively, $T(\theta) = (\sin \theta, \cos \theta)$) is inward normal vector (respectively, tangent vector).

We point out two basic properties of solutions of Problem 1.1. First, the motion by a power of curvature is a curve-shortening motion, since

\begin{equation}
\frac{d}{dt} L(t) = \frac{d}{dt} \int_{\Gamma(t)} ds = \int_0^{2\eta\pi} \frac{\partial}{\partial \theta} V^{-\frac{1}{2}} d\theta = - \int_{\Gamma(t)} K^{\alpha+1} ds < 0,
\end{equation}

where we denote the arc-length of $\Gamma(t)$ by $L(t)$. $\int ds$ denotes an integration by the arc-length parameter on $\Gamma(t)$. One can further show that the length of the curve decreases "pointwise" in a certain sense (see (2.25) in [9]). Next, the maximum in $\theta$ of solution $V(\theta, t)$ goes to infinity in a finite time $T_*$. If $\Gamma(t)$ is an embedded curve, then it is known that $\Gamma(t)$ shrinks to a point as $t \nearrow T_*$ [5], [8]. But this is not necessarily the case if $\Gamma(t)$ has self-intersection. In fact, the curvature blows up if some of the small "loops" on $\Gamma(t)$ shrink and disappear. The time

$$T_* := \sup \left\{ t \mid \max_{\theta \in T_\eta} V(\theta, t) < \infty \right\}$$

is called the blow-up time.

It may also be interesting to note that the behavior of shrinking curves exhibits a dramatic contrast between the case $\alpha > \frac{1}{2}$ and the case $0 < \alpha < \frac{1}{2}$. More precisely, when $\alpha > \frac{1}{2}$, any embedded closed convex curve eventually approaches a shrinking
circle. This fact was first discovered by Gage–Hamilton [5] and later extended to the general case $\alpha > \frac{1}{3}$ by Taniyama–Matano [19]. On the other hand, if $0 < \alpha < \frac{1}{3}$, then there are solutions whose curvature blows up at a faster rate than a shrinking circle (Type II blow-up) and whose aspect ratio tends to infinity as it shrinks to a point. See Taniyama–Matano [19] for details. Matano–Wei [13] generalizes this result for an anisotropic equation.

Let us explain the crystalline algorithm. Some materials have nonsmooth interfacial energies. To study the motion of interfaces under these nonsmooth interfacial energies, Taylor [20] and Angenent–Gurtin [2], independently, introduced crystalline energies. For these energies, the corresponding solutions are polygonal and the evolution law is a system of ordinary differential equations. For the convex closed piecewise linear curve, which consists of $n$ sides and whose sides intersect at the vertices $\{B_j(t)\}_{0 \leq j < n}$ with the angle $\pi - \Delta \theta$ ($\Delta \theta = \frac{2\pi}{n}$), the evolution is described by

$$\frac{d}{dt} d_j(t) = -(v_{j+1}(t) \csc \Delta \theta - 2v_j(t) \cot \Delta \theta + v_{j-1}(t) \csc \Delta \theta)$$

(1.6)

(see [2, (10.18)], [21], and references therein), where $d_j(t) := |B_{j+1}(t) - B_j(t)|$ is the length of the $j$th side of this piecewise linear curve. This equation is derived from space discretized version of (1.3),

$$\frac{d}{dt} X_j(t) = v_j(t) N_j, \quad 0 \leq j < n, \quad t > 0,$$

(1.7)

and the relation between $\{X_j(t)\}_{0 \leq j < n}$ and $\{B_j(t)\}_{0 \leq j < n}$,

$$B_j(t) = (X_{j-1}(t) - X_j(t)) \cdot (T_j + N_j \cot \Delta \theta) T_j + X_j(t),$$

where $N_j := N(j\Delta \theta)$ (respectively, $T_j := T(j\Delta \theta)$) is the interior normal vector (respectively, tangent vector) to the $j$th side.

Moreover, Taylor and Angenent–Gurtin introduced crystalline curvature. Crystalline curvature for the $j$th side of the closed convex piecewise linear curve is given by

$$k_j := \frac{2\tan \frac{\Delta \theta}{2}}{d_j}.$$

(1.8)

Let us make the relation between the usual curvature and the above quantity clear. If the curvature is defined in the standard way (the derivative of the angle between the tangent and a fixed axis with respect to the arc-length), then the curvature of piecewise linear curve is zero at each side and not defined at each vertex of the curve. However, if we define curvature by a derivative of the length with respect to a signed area for smooth deformations of the curve, then for a smooth curve it coincides with usual curvature and for a piecewise linear curve it coincides with crystalline curvature (see [6], [17]). Using (1.6), (1.7), (1.8), and the evolution law, one can analyze the curvature motions of piecewise linear curves.

This idea is used not only for analysis, but also for numerical simulation. In fact, when we approximate smooth curves by piecewise linear curves and curvature by crystalline curvature, we can obtain approximate solution by integrating (1.6), (1.7), (1.8), and the evolution law. We call the numerical algorithm employing this crystalline approximation crystalline algorithm (see section 2). Several authors have studied convergence of the crystalline algorithm. For the motion by weighted mean
curvature, convergence has been proved by [4], [7] for graphs, by [6] for simple closed convex curves, and by [10] for nonconvex closed curves. In these papers, discretization in time is not considered.

In this paper, we apply the crystalline algorithm to the case of motion by a power of curvature. For (1.6) and (1.7) time discretization is necessary for numerical computation. We use the Euler method for the time discretization, namely,

\[ d_j^{m+1} = d_j^m - (v_j^m \csc \Delta \theta - 2v_j^m \cot \Delta \theta + v_{j-1}^m \csc \Delta \theta) \tau_{m+1}, \quad m = 0, 1, 2, \ldots, \]

and

\[ X_j^{m+1} = X_j^m + v_j^m N_j \tau_{m+1}, \quad m = 0, 1, 2, \ldots, \]

where \( d_j^m \) is the length of \( j \)th side in the approximate piecewise linear curve for \( \Gamma(t_m) \), \( v_j^m \) is approximation of \( V(j \Delta \theta, t_m) \), \( \tau_{m+1} \) is a variable time difference and \( t_m \) is time. By rearranging the above equations a full discretized problem for Problem 1.1 is obtained. Applying a fundamental comparison principle to this problem, we prove the convergence of our scheme.

In our scheme, we control the time step \( \tau_{m+1} \) by choosing

\[ \tau_{m+1} = \frac{\lambda_\alpha(\varepsilon) \Delta \theta^2}{(v_m^{\max})^{\alpha}}, \quad \lambda_\alpha(\varepsilon) := \frac{1}{2(1 + \varepsilon)\alpha + 1} \max\{1, \alpha\}, \]

where \( \varepsilon \) is any fixed positive number. See Procedure 2.2 for details. Here and hereafter, we set \( (\cdot)^m_{\max} := \max_{0 \leq j < n} (\cdot)^m_j \). We set

\[ \|V\|_T := \sum_{k=0}^4 \sup_{0 \leq t \leq T, \theta \in T_n} \left| \frac{\partial^k}{\partial \theta^k} V(\theta, t) \right| + \sum_{k=1}^2 \sup_{0 \leq t \leq T, \theta \in T_n} \left| \frac{\partial^k}{\partial \theta^k} V(\theta, t) \right| \]

and

\[ \|V\|_0 := \sum_{k=0}^2 \sup_{\theta \in T_n} \left| \frac{\partial^k}{\partial \theta^k} V(\theta, 0) \right|. \]

We define that \( (\Gamma(t), V(\theta, t)) \) is a solution of (1.1) if and only if \( \{\Gamma(t)\}_{t \geq 0} \) is a smooth, closed, convex, and immersed curve which moves according to (1.1) with normal velocity \( \{V(\theta, t)\}_{t \geq 0} \). We also define that \( (\Gamma^m, v_j^m) \) is a solution of our scheme if and only if \( \{\Gamma^m(t)\}_{m \geq 0} \) is a piecewise linear, closed, and convex curve which moves according to our scheme with normal velocity \( \{v_j^m\}_{0 \leq j < n, m \geq 0} \).

Now we state our main results.

**Theorem 1.2.** Fix a parameter \( \alpha > 0 \) and a winding number \( \eta \geq 1 \). Let \( (\Gamma(t), V(\theta, t)) \) be a solution of (1.1) and \( T_* \) the blow-up time of this solution. Fix a positive number \( \varepsilon \). Let \( (\Gamma^m, v_j^m) \) be the solution of our scheme with the time step control above. For any \( T \in [0, T_*) \), there exist finite constants \( C_v, C_d, \) and \( C_l \) and positive integers \( n_v, n_d, \) and \( n_l \) such that the following error estimates hold:

\[
\begin{align*}
(a) \quad & \max_{0 \leq j < n} |v_j^m - V(j \Delta \theta, t_m)| \leq C_v(\Delta \theta)^2 \text{ for all } n \geq n_v, 0 \leq t_m \leq T, \\
(b) \quad & D_H(\Gamma^m, \Gamma(t_m)) \leq C_d(\Delta \theta)^2 \text{ for all } n \geq n_d, 0 \leq t_m \leq T, \\
(c) \quad & |L^m_m - L(t_m)| \leq C_l(\Delta \theta)^2 \text{ for all } n \geq n_l, 0 \leq t_m \leq T,
\end{align*}
\]
where constants $C_v, C_d, C_l$ depend only on $T$, $\min_{\theta \in T} \varphi(\theta)$, and $\|V\|_T$ and integers $n_v, n_d, n_l$ depend on $T$, $\|V\|_T$, $\min_{\theta \in T} \varphi(\theta)$, and $\varepsilon$.

Here $t_m := \sum_{i=1}^{m} \tau_i$ is time, $D_H$ is the Hausdorff distance, and $L_m^\Delta$ (respectively, $L(t_m)$) is the length of $\Gamma_m^\Delta$ (respectively, the arc-length of $\Gamma(t_m)$).

When $\eta = 1$, we set that $A_m^\Delta$ (respectively, $A(t_m)$) is the area of the region enclosed by $\Gamma_m^\Delta$ (respectively, $\Gamma(t_m)$). The following corollary holds.

**Corollary 1.3.** Suppose that the same assumptions in the convergence theorem hold. Then there exist positive constant $C_a$ and natural number $n_a$ such that the following estimate is satisfied:

$$|A_m^\Delta - A(t_m)| \leq C_a (\Delta \theta)^2$$

for all $n \geq n_a$, where constant $C_a$ depends on $T, \min_{\theta \in T} \varphi(\theta)$ and $\|V\|_T$, and integer $n_a$ depends on $T, \min_{\theta \in T} \varphi(\theta), \|V\|_T$, and $\varepsilon$.

**Remark 1.4.** Natural numbers $n_v$, etc., and positive $\varepsilon > 0$ satisfy the inequality $n_v, n_d, n_l, n_a \geq \frac{C}{\sqrt{\varepsilon}}$. Here constant $C = C(T, \|V\|_T, \min_{\theta \in T} \varphi(\theta))$ does not depend on $\varepsilon$.

The positivity of $\varepsilon$ leads to the well-posedness of our scheme (see Lemma 2.7). Moreover, the comparison principle is available due to the positivity of $\varepsilon$ (see the proof of Theorem 1.2(a)).

**Remark 1.5.** The error estimates need high regularity of the continuous solution. However, our solutions do have such regularity, since smooth local solutions exist for Problem 1.1.

**Remark 1.6.** The total length of solution of our scheme is monotone non-increasing (see Remark 3.2 and the proof of Theorem 3.4). It corresponds to the fact that Problem 1.1 is a curve-shortening equation. However, the length of each side $d_m^j$ is not necessarily nonincreasing.

In papers [11] and [12], M. Kimura establishes a scheme for computing curves which are evolved by mean curvature flow ($\alpha = 1$) and which show the convergence of his scheme. In the case when $\alpha = 1$ and the initial curve $\Gamma_0$ is embedded in $\mathbb{R}^2$ (i.e., $\eta = 1$), it is well known that the $\Gamma(t)$ shrinks to a “round point” in finite time: its asymptotic shape just before it disappears is a circle [5], [8]. In the Kimura’s scheme tracking points are kept uniformly distributed on the arc-length of curves, then his scheme fits on tracking the curve when $\alpha = 1$.

The recent papers [14] and [15], consider a problem similar to Problem 1.1 and present a numerical scheme. Their scheme is based on Rothe’s approximation in time and on the finite element approach in space.

The organization of this paper is as follows: in section 2, we give a crystalline algorithm for our problem. The rate of convergence for the initial data and some properties of our scheme are also shown. In section 3, proofs of Theorem 1.2 and Corollary 1.3 are given, and some numerical examples are shown in section 4.

**2. Crystalline algorithm.** In this section we explain our scheme for computing solutions of (1.1). We also give some properties of this scheme.

Let $\Gamma_0$ be a smooth, strictly convex, immersed and closed curve with winding number $\eta \geq 1$. We approximate $\Gamma_0$ by a piecewise linear curve with $n$ line segments. We set

$$\Delta \theta := \frac{2\eta \pi}{n}, \quad n \geq 4\eta, \quad \eta = 1, 2, \ldots,$$
and so $0 < \Delta \theta \leq \frac{\pi}{2}$. For the sake of convenience, we set

$$\Delta \theta_2 := 2(1 - \cos \Delta \theta) = (\Delta \theta)^2 - \frac{1}{12}(\Delta \theta)^4 + \cdots,$$

and so $0 < \Delta \theta_2 \leq 2$.

The way of crystalline approximation for $\Gamma_0$ is as follows (see also Figure 2.1).

**Procedure 2.1** (crystalline approximation for $\Gamma_0$). We denote the subscript number by $j = 0, 1, \ldots, n - 1$. Let $X_0^j$ be the point on $\Gamma_0$ with inward normal vector $N_j := -(\cos j \Delta \theta, \sin j \Delta \theta)$. The tangent vector at $X_0^j$ is $T_j := (-\sin j \Delta \theta, \cos j \Delta \theta)$.

Let $\Gamma_0^\Delta$ be the union of line segments $B_0^j B_0^{j+1}$ where $B_0^j$ is the intersection point between the lines tangent to $\Gamma_0$ at $X_0^{j-1}$ and $X_0^j$ ($B_0^n = B_0^0$).

The line segment $B_0^j B_0^{j+1}$ is called the $j$th side and the length of the $j$th side is denoted by $d_0^j$. The curvature of the $j$th side is defined by

$$k_0^j := \frac{2\tan \frac{\Delta \theta}{2}}{d_0^j}$$

(see (45) of [7]) and the velocity is proposed by $v_0^j := (k_0^j)^\alpha$.

Fix a positive number $\varepsilon$. Now we state our algorithm.

**Procedure 2.2** (crystalline algorithm). For given $\{v_m^j\}_{0 \leq j < n}$, $\{X_m^j\}_{0 \leq j < n}$ and $\Gamma_m^\Delta$, we define $\{v_{m+1}^j\}_{0 \leq j < n}$, $\{X_{m+1}^j\}_{0 \leq j < n}$ and $\Gamma_{m+1}^\Delta$ as follows:

(a) the time step,

$$\tau_{m+1} := \frac{\lambda_\alpha(\varepsilon)\Delta \theta_2}{(v_m^\text{max})^{\alpha + 1}}, \quad \lambda_\alpha(\varepsilon) := \frac{1}{2(1 + \varepsilon)^{\alpha + 1}\max\{1, \alpha\}}, \quad \varepsilon > 0;$$

(b) the $j$th normal velocity,

$$v_{m+1}^j := (1 - (v_m^j)^{\frac{1}{\alpha}}(\Delta \theta^2 v + v_m^j)^{\tau_{m+1}})^{-\alpha};$$

(c) the $j$th tracking point,

$$X_{m+1}^j := X_m^j + v_m^j N_j \tau_{m+1};$$
(d) the $j$th vertex $B_{j}^{m+1}$ of $\Gamma_{\Delta}^{m+1}$,

$$B_{j}^{m+1} := \left( \left( X_{j-1}^{m+1} - X_{j}^{m+1} \right) \cdot (T_{j} + N_{j} \cot \Delta \theta) \right) T_{j} + X_{j}^{m+1};$$

where $(\Delta_{\theta}(\cdot))_{j}^{m} := \frac{1}{\Delta \theta_{2}} \left( \frac{1}{\Delta \theta_{2}} - 2 \cdot \left( \frac{1}{\Delta \theta_{2}} + \frac{1}{\Delta \theta_{2}} \right) \right)$ is a kind of central difference approximation to $\frac{\partial^{2} \varphi}{\partial x^{2}}$ at $(j \Delta \theta, t_{m})$, piecewise linear curve $\Gamma_{\Delta}^{m}$ is the union of the line segments $B_{j}^{m} B_{j+1}^{m}$ which is approximation of the curve $\Gamma(t_{m})$, and $\Delta := \sum_{i=1}^{m} \Delta_{i}$ is "time."

Due to the definition of crystalline curvature (1.8) and the evolution law (1.1), Procedure 2.2(b) is readily obtained from (1.9).

We note that the $0$th side always intersects with $x$-axis orthogonally since $N_{0} \equiv (-1,0)$.

**Remark 2.3.** Because of the explicit time discretization and the variable time step, our algorithm may look expensive for a one-dimensional problem. However, as we are dealing with solutions that blow up in finite time $T_{\ast}$, where $T_{\ast}$ is not known a priori, the use of varying time step is unavoidable. As for the efficiency of implicit time discretization in comparison with the explicit one, we are currently making a detailed research which will appear in a forthcoming paper.

The next estimate is obtained by Girão [6].

**Proposition 2.4** (see [6]). For all winding number $\eta \geq 1$, there exist positive constant $C_{k}^{0} = C_{k}^{0}(\|K\|_{0}, \min_{\theta \in T_{\ast}} \varphi(\theta))$ and natural number $n_{k}^{0} > 0$ such that for all $n \geq n_{k}^{0}$

$$\max_{0 \leq j < n} |k_{j}^{0} - K(j \Delta \theta, 0)| \leq C_{k}^{0} \Delta \theta_{2}.$$ 

The following lemma shall show that Theorem 1.2(a,c) and its Corollary 1.3 hold for $m = 0$. Proof of Theorem 1.2(b) for $m = 0$ is given in section 3. We denote the arc-length of $\Gamma_{\Delta}^{0}$ by $L_{\Delta}^{0} = \sum_{j=0}^{n-1} \theta_{j}$. The next lemma and the above proposition ensure that the initial curves can be approximated through Procedure 2.1.

**Lemma 2.5.** For all $\alpha > 0$ and $\eta \geq 1$ there exist positive constants $C_{\alpha}^{0}, C_{l}^{0}$ and natural numbers $n_{\alpha}^{0}, n_{l}^{0} > 0$ such that

$$\max_{0 \leq j < n} |v_{j}^{0} - V(j \Delta \theta, 0)| \leq C_{\alpha}^{0} (\Delta \theta)^{2} \quad \text{for all} \quad n \geq n_{\alpha}^{0}$$

and

$$|L_{\Delta}^{0} - L(0)| \leq C_{l}^{0} (\Delta \theta)^{2} \quad \text{for all} \quad n \geq n_{l}^{0}.$$ 

Moreover if $\eta = 1$, then there exist positive constant $C_{\alpha}^{0}$ and natural number $n_{\alpha}^{0}$ such that for all $n \geq n_{\alpha}^{0}$,

$$|A_{\Delta}^{0} - A(0)| \leq C_{\alpha}^{0} (\Delta \theta)^{2}.$$ 

Here all constants depend only on $\|K\|_{0}$ and $\min_{\theta \in T_{\ast}} \varphi(\theta)$.

*Proof.* Here and hereafter, we use the notation $\min_{\theta \in T_{\ast}} \varphi(\cdot)$ (respectively, $\max_{\theta \in T_{\ast}} \varphi(\cdot)$) for $\min_{\theta \in T_{\ast}} \varphi(\cdot)$ (respectively, $\max_{\theta \in T_{\ast}} \varphi(\cdot)$).

**Proof of (2.1).** By Proposition 2.4, we have $K(j \Delta \theta, 0) - C_{k}^{0} \Delta \theta_{2} \leq k_{j}^{0} \leq k(j \Delta \theta, 0) + C_{k}^{0} \Delta \theta_{2}$. Then

$$V(j \Delta \theta, 0) \left( 1 - \frac{C_{k}^{0}}{K(j \Delta \theta, 0)} \Delta \theta_{2} \right)^{\alpha} \leq v_{j}^{0} \leq V(j \Delta \theta, 0) \left( 1 + \frac{C_{k}^{0}}{K(j \Delta \theta, 0)} \Delta \theta_{2} \right)^{\alpha}$$
for all \( n \geq n_0^0 \), where \( n_0^0 := \min \left\{ n \geq n_k^0 \mid \frac{2c_k^0}{\min_k K(\theta,0)} \Delta \theta_2 \leq 1 \right\} \). Therefore,

\[
V(j \Delta \theta, 0) - \frac{\alpha C_k^0 V(j \Delta \theta, 0)}{K(j \Delta \theta, 0)} \left( 1 - \frac{(\alpha - 1)(1 + \xi_0^+)^\alpha C_k^0 \Delta \theta_2}{2(1 + \xi_0^+)^2 K(j \Delta \theta, 0)} \right) \Delta \theta_2 \\
\leq v_j^0 = (k_j^0)^\alpha \\
\leq V(j \Delta \theta, 0) + \frac{\alpha C_k^0 V(j \Delta \theta, 0)}{K(j \Delta \theta, 0)} \left( 1 + \frac{(\alpha - 1)(1 + \xi_0^+)^\alpha C_k^0 \Delta \theta_2}{2(1 + \xi_0^+)^2 K(j \Delta \theta, 0)} \right) \Delta \theta_2,
\]

where \( |\xi_0^+| \leq \frac{c_k^0}{K(j \Delta \theta, 0)} \Delta \theta_2 \leq \frac{1}{2} \) are the suitable values on the mean value theorem. Hence if we set

\[
C_v^0 := \frac{\alpha C_k^0 \max_\theta V(\theta,0)}{\min_\theta K(\theta,0)} \left( 1 + |\alpha - 1| 2^{-a} 3^\alpha \frac{C_k^0}{\min_k K(\theta,0)} \right),
\]

then the assertion holds for all \( n \geq n_0^0 \). \( \Box \)

**Proof of (2.2).** After brief calculation, we have

\[
L(0) = \int_0^{2\pi} \frac{d\theta}{K(\theta,0)} = \sum_{j=0}^{n_k^0 - 1} \int_{(j + \frac{1}{2}) \Delta \theta}^{(j + \frac{1}{2}) \Delta \theta + \Delta \theta} \frac{d\theta}{K(\theta,0)} = \sum_{j=0}^{n_k^0 - 1} \frac{1}{K(j \Delta \theta, 0)} \left( \Delta \theta + \frac{1}{12} \xi_\alpha(\Delta \theta)^3 \right)
\]

for \( n \geq n_k^0 \), where \( \xi_\alpha \) is a suitable value on the mean value theorem and

\[
n_k^0 := \min \left\{ n \geq n_k^0 \mid \frac{\Delta \theta}{2 \min_k K(\theta,0)} \left( \max_\theta K_\beta(\theta,0) + \frac{\Delta \theta}{4} \max_\theta K_{\beta\theta}(\theta,0) \right) < 1 \right\}.
\]

By mean value theorem, \( 2 \tan \frac{\Delta \theta}{2} = \Delta \theta + \xi_\tan(\Delta \theta)^3 \) for a suitable value \( \xi_\tan \). Hence there exists \( C_1^0 > 0 \) such that

\[
|L_\alpha^0 - L(0)| \leq \sum_{j=0}^{n_k^0 - 1} \frac{2 \tan \frac{\Delta \theta}{2}}{k_j^0} - \frac{\Delta \theta + \xi_\tan(\Delta \theta)^3}{12} K(j \Delta \theta, 0)
\]

\[
\leq \left( \left| C_k^0 \frac{\Delta \theta_2}{(\Delta \theta)^2} \right| + \left| \frac{\xi_\tan K(j \Delta \theta, 0) - \xi_\tan K(j \Delta \theta, 0)_{\max}}{12} \sum_j \frac{\Delta \theta}{k_j^0 K(j \Delta \theta, 0)} \right| \right) (\Delta \theta)^2
\]

\[
\leq C_1^0 (\Delta \theta)^2,
\]

where we denote \( |\cdot|_{\max} := \max_0 \leq j < n |(\cdot)|_j \). Therefore the assertion holds for all \( n \geq n_k^0 \). \( \Box \)

**Proof of (2.3).** Here and hereafter, we use the notation \( \cdot_{\min} \) and \( \cdot_{\max} \) for \( \min_0 \leq j < n (\cdot)_j \) and \( \max_0 \leq j < n (\cdot)_j \), respectively.

It can be shown that there exists \( \xi_\theta > 0 \) such that for all \( n \geq n_0^0 := n_v^0 \)

\[
d_n^0 \max = \frac{2 \tan \frac{\Delta \theta}{2}}{k_n^0 \min} \leq \left( \frac{1 + \xi_\tan(\Delta \theta)^2}{\min_k K(\theta,0) - C_k^0 \Delta \theta_2} \right) \Delta \theta \leq \xi_\theta \Delta \theta,
\]

where \( n_0^0 \) and \( \xi_\tan \) be the same as the above. Since the difference between \( A_\alpha^0 \) and \( A(0) \) is less than the sum of the area of triangle \( X_j^0 B_{j+1}^0 X_{j+1}^0 \), there exists \( C_a^0 > 0 \) such that

\[
0 < A_\alpha^0 - A(0) < \frac{1}{2} \sum_j |B_{j+1}^0 - X_j^0||X_{j+1}^0 - B_{j+1}^0| \sin \Delta \theta
\]

\[
\leq \frac{1}{2} \sum_j d_j^0 d_{j+1}^0 \sin \Delta \theta \leq C_a^0 (\Delta \theta)^2
\]
for all \( n \geq n_0^\eta \).

\[ \square \square \]

The winding number of \( \Gamma_\Delta^m \) does not change through Procedure 2.2. In other words, nodes do not “cross over.”

**Lemma 2.6.** For every step \( m \), the angle between two adjacent sides of \( \Gamma_\Delta^m \) is \( \pi - \Delta \theta \).

**Proof.** It is sufficient to show that the vector \( B_{j+1}^m - B_j^m \) is parallel and equidirectional to the tangent vector \( T_j \), that is,

\[(3.1c) \quad (B_{j+1}^m - B_j^m) \cdot T_j \equiv d_j^m \]

for \( j = 0, 1, \ldots, n - 1 \).

From Procedure 2.2(d) and (1.9), we have

\[
B_{j+1}^m - B_j^m = ((X_{j+1}^m - X_{j-1}^m) \cdot T_j - (X_{j+1}^m - 2X_j^m + X_{j-1}^m) \cdot N_j \cot \Delta \theta) T_j
= (d_j^m - (v_j^m - 2v_j^m \cos \Delta \theta + v_{j-1}^m) \frac{\tau_{m+1}}{\sin \Delta \theta}) T_j
eq \hat{d}_j T_j,
\]

here \( X_j^m = X_j^m + v_j^m N_j \tau_{m+1} \) are substituted. Hence

\[
\hat{d}_j \geq d_j^m - \frac{2\lambda_\alpha(\varepsilon)v_{\max}^m}{(v_{\max}^m)^{\frac{\alpha-1}{\alpha}}} \frac{\Delta \theta_2 \geq d_j^m - 2\lambda_\alpha(\varepsilon)}{k_{\max}^m} > d_j^m - d_{\min}^m \geq 0
\]

since \( \lambda_\alpha(\varepsilon) < \frac{1}{2} \) and so \( \hat{d}_j = d_j^{m+1} > 0 \).

In this proof, the positivity of \( \{d_j^m\}_{0 \leq j < n, m \geq 0} \) is important for well-posedness of our scheme so we state this property as a lemma.

**Lemma 2.7.** For all \( \eta \geq 1, \alpha > 0 \) and \( n \geq 4\eta \), no sides of \( \Gamma_\Delta^m \) vanish for a finite step \( m \).

**Remark 2.8.** The previous lemma guarantees that solution of our scheme is successively determined up to any finite step by Procedure 2.2. In other words, \( v_{\max}^m \) does not blow up for all \( 0 \leq t_m < T_\Delta \), where \( T_\Delta := \sum_{i=1}^{\infty} \tau_i \).

**3. Proof of Theorem 1.2 and Corollary 1.3.** In this section we prove our main theorem and its corollary. At first, we introduce Problem 3.1 below. It can be readily observed that solutions of our scheme must satisfy this problem.

**Problem 3.1 (full discretized problem).** Find a sequence \( \{v_j^m\}_{0 \leq j < n, m \geq 0} \) satisfying

\[
(3.1a) \quad -(D_\tau v^m) = (\Delta \theta v + v)_j^m, \quad m = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, n - 1,
(3.1b) \quad v_j^0 = (k_j^0)^\alpha, \quad j = 0, 1, \ldots, n - 1,
(3.1c) \quad v_0^m = v_n^m, \quad v_{n-1}^m = v_{n-1}, \quad m = 0, 1, 2, \ldots,
\]

where

\[
(3.1d) \quad (\Delta \theta v)_j^m := \frac{v_{j+1}^m - v_{j-1}^m}{\Delta \theta_2}, \quad \Delta \theta_2 = 2(1 - \cos \Delta \theta)
\]

is a kind of central difference approximation to \( V_{\theta}(j \Delta \theta, t_m) \),

\[
(D_\tau (\cdot))_j^m := \frac{(\cdot)_j^{m+1} - (\cdot)_j^m}{\tau_{m+1}}
\]
is the time difference, the time-step $\tau_{m+1}$ is the same as in Procedure 2.2 and the crystalline curvature \{\kappa_t^0\}_{0\leq t\leq n} is provided in Procedure 2.1.

Remark 3.2. This way of discretization is somehow straight-forward from the original continuous problem (Problem 1.1), when we expect that the following properties still hold for obtaining discretized problem:
1. Curve-shortening property (see (1.5)).
2. Closedness of $\Gamma_n^m$. If $\Gamma_n^m$ is closed, then $\Gamma_n^m$ is closed for all step $m$. More precisely, closedness condition which is corresponding to (1.2d) is satisfied for all step $m$.

From property 1, we should discretize the left-hand side of the partial differential equation $-(V - \frac{\partial \tau}{\partial t}) = V\theta + V$ (which is equivalent to (1.2a)) by $-(D_r V - \frac{\partial \tau}{\partial t})^m$. Property 2 suggests the way of discretization of Laplacian $\frac{\partial^2 \tau}{\partial \theta^2}$ should be (3.1d). Conversely, $\Gamma_n^m$ is automatically closed for all $\eta \geq 1$, $\alpha > 0$ and $n \geq 4\eta$. That is, $\sum_{j=0}^{n-1} e_j^m T_j \equiv 0$ holds for all step $m = 0, 1, 2, \ldots$ if the approximation of Laplacian is given by (3.1d).

To prove our main theorem, we repeatedly use the following fundamental comparison lemma which is a result of an easy calculation.

Lemma 3.3 (comparison principle). Let $F : \mathbb{R}^n \to \mathbb{R}^n; (x_j) \mapsto (F_j(x))$ be a $C^1$-mapping. Assume that $U, V \in \mathbb{R}^n$ satisfy

$$U \geq V.$$

If each element of Jacobi matrix is nonnegative, that is,

$$\frac{\partial F_i}{\partial x_j}(sU + (1 - s)V) \geq 0,$$

for all $0 \leq i, j < n$, $0 \leq s \leq 1$, then $F(U) \geq F(V)$ holds.

Here $U \geq V$ means that each element of $U$ is greater than or equal to the corresponding element of $V$.

In order to apply this comparison lemma to Problem 3.1 we rewrite this problem in vector form. For this purpose it is convenient to use the equations for crystalline curvature, that is,

$$\begin{array}{l}
(D_r k)^m_j = k_{j+1}^m k_j^m (\Delta \theta + k^\alpha)^m_j, \quad m = 0, 1, 2, \ldots, \\ j = 0, 1, \ldots, n - 1,
\end{array}$$

where we set $(k_\alpha^m)^j = (k_j^m)^\alpha$. Clearly, these equations are equivalent to the equation (3.1a). We define nonlinear operator $F^m = (F_0^m, \ldots, F_{n-1}^m) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_j^m(x) = x_j (1 - x_j (\Delta \theta + x^\alpha)_{j+1} \tau_{m+1})^{-1}, \quad x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n.$$

One can easily check that equations (3.3) are expressed as

$$k^{m+1} = F^m(k^m),$$

where $k^m = (k_0^m, \ldots, k_{n-1}^m)$.

We note that Jacobi matrix of $F^m$ is

$$\left(\frac{\partial F^m_i(x)}{\partial x_j}\right)_{i,j} = \left(\frac{\partial F^m_i(x)}{\partial x_j}\right),$$

(3.4)

$$= \begin{cases}
\left(\frac{F_i^m(x)}{x_i}\right)^2 \left(1 - 2\alpha x_i^{x_{i+1}^\alpha} \cos \Delta \theta \frac{\tau_{m+1}}{\Delta \theta_2}\right)^2, & j = i, \\
\alpha \frac{F_i^m(x)}{x_i} \Delta \theta_2, & j = i + 1, \\
0, & \text{otherwise}
\end{cases}$$
for $0 \leq i, j < n$, where $(\cdot)_{i,j}$ is a $(i,j)$-entry of matrix $(\cdot)$.

In this section we shall prove a blow-up result and the convergence theorem using a discrete analogue of supersubsolution method. First we note that for a given sequence $\{\tau_{m+1}\}_{m \geq 0}$, a sequence of operators $\{F^m\}_{m \geq 0}$ is determined. Fix sequences $\{\tau_{m+1}\}_{m \geq 0}$ and $\{F^m\}_{m \geq 0}$. We call sequences $\{k^m\}$ and $\{l^m\}$ supersolution and subsolution, respectively, if they satisfy

$$k^{m+1} \geq F^m(k^m) \quad \text{(3.5)}$$

and

$$l^{m+1} \leq F^m(l^m). \quad \text{(3.6)}$$

Let $\{k^m\}$ be a supersolution and $\{l^m\}$ a subsolution. If for $m = m_0$ we have $k^{m_0} \geq l^{m_0}$ and (3.2) with $U = k^{m_0}$ and $V = l^{m_0}$, then by Lemma 3.3 we immediately obtain

$$k^{m_0 + 1} \geq l^{m_0 + 1}.$$ 

We note that conditions (3.5) and (3.6) are equivalent to

$$(D_{\tau}k)^m_j \geq k^{m+1}_j (\Delta \theta k^m + k^m_j) \quad \text{and} \quad (D_{\tau}l)^m_j \leq k^{m+1}_j (\Delta \theta k^m + k^m_j),$$

respectively, where we set $k^m = (k^m_j)_{0 \leq j < n}$ and $l^m = (l^m_j)_{0 \leq j < n}$. One easily sees that these conditions are a discretized analogue of the conditions for super- and subsolutions for Problem 1.1, respectively.

Hereafter we omit the suffix $m$ of operator $F^m$.

**Theorem 3.4** (blow-up of solutions for Problem 3.1). Fix a winding number $\eta \geq 1$, a parameter $\alpha > 0$, and a partition number $n$ such that $n \geq 4 \eta$. The solution of Problem 3.1 $v^m_j$ blows up in finite time, that is,

$$T_{\Delta} := \sum_{i=1}^{\infty} \tau_i < \infty$$

and

$$\lim_{m \to T_{\Delta}} \max_{0 \leq j < n} v^m_j = \lim_{m \to \infty} \max_{0 \leq j < n} v^m_j = \infty.$$ 

**Proof.** Let $v^m_j$ be a solution of Problem 3.1. We set $k^m_j = (v^m_j)^{\frac{1}{2}}$, $k^m := \{k^m_j\}$, $k^{m}_{min} := \min_{0 \leq j < n} k^m_j$ and $k^{m}_{max} := \{k^{m}_{min}\}$. Apparently we have

$$k^m \geq k^{m}_{max} \quad \text{and} \quad F(k^{m}_{max}) \geq k^{m}_{min}.$$ 

Moreover, taking our time-step control into account, we can check that the condition (3.2) holds with $U = k^m$ and $V = k^{m}_{min}$. Hence by Lemma 3.3, we obtain

$$k^{m+1} = F(k^m) \geq F(k^{m}_{min}) \geq k^{m}_{min}.$$ 

Therefore $k^{m}_{min}$ is monotone increasing with respect to $m$. Hence we have $k^m \geq k^0_{min}$. 

We note the relation between the length of $\Gamma^m_\Delta$ and the crystalline curvature $k^m_j$:
\[
L^m_\Delta = \sum_{j=0}^{n-1} d^m_j = \sum_{j=0}^{n-1} \frac{2 \tan \frac{\Delta \theta}{2}}{k^m_j}.
\]
Calculating just as (1.5), we obtain
\[
(D_r L_\Delta)^m = -\sum_{j=0}^{n-1} 2 \tan \frac{\Delta \theta}{2} (k^m_j)\alpha \leq -2 \eta \pi (k^0_{\min})^\alpha.
\]
For any fixed integer $M$, we have
\[
L^M_\Delta - L^0_\Delta = \sum_{m=0}^{M} (D_r L_\Delta)^m \tau_{m+1} \leq -2 \eta \pi (k^0_{\min})^\alpha \sum_{m=0}^{M} \tau_{m+1} = -2 \eta \pi (k^0_{\min})^\alpha t_M
\]
and $t_M \leq \frac{L^0_\Delta}{2 \eta \pi (k^0_{\min})^\alpha}$. Thus $T_\Delta < \infty$. Therefore $\lim_{m \to \infty} \tau_m = 0$, and the result is proved.

Now we prove Theorem 1.2 and Corollary 1.3.

**Proof of Theorem 1.2.** First we prove convergence of curvature. Next we show the convergence of curves in Hausdorff distance. Proof of Theorem 1.2(c) is quite parallel to proof of (2.2) in Lemma 2.5, so it can safely be omitted.

**Proof of convergence of curvature and Theorem 1.2(a).** Let $V(\theta, t)$ be a solution of Problem 1.1 and we set $K^m_j := V(j \Delta \theta, t_m)^\frac{1}{\Delta}$. For convenience. We define sequences $\overline{K}^m_j, \underline{K}^m_j$ as follows:
\[
\overline{K}^m_j := K^m_j + A e^{B_m} \Delta \theta_2 \quad \text{and} \quad \underline{K}^m_j := K^m_j - A e^{B_m} \Delta \theta_2.
\]
We set $\overline{K}^m := (\overline{K}^m_0, \ldots, \overline{K}^m_{n-1})$ and $\underline{K}^m := (\underline{K}^m_0, \ldots, \underline{K}^m_{n-1})$. It can be shown that there exist $A, B, \Delta \theta_2$ such that $\overline{K}^{m+1}_j$ and $\underline{K}^{m}_j$ (respectively, $\overline{K}^{m+1}_j$ and $\underline{K}^{m}_j$) satisfy (3.5) (respectively, (3.6)) for $\Delta \theta \leq \Delta \theta_2$ and $m \geq 0$. In fact, after calculations in a few lines, we have
\[
(D_r \overline{K}^m_j)^m = \overline{K}^m_j \overline{K}^{m+1}_j \left( \Delta \theta \overline{K}^m_j + \underline{K}^m_j \right)^m
\]
\[
\quad + \left( A e^{B_m} (B - a^m_j e^{B_{m+1} - b^m_j}) + K^m_j K^{m+1}_j c^m_j \right) \Delta \theta_2 + \text{h.o.t.,}
\]
where $a^m_j, b^m_j, c^m_j$ are some constants which only depend on $T$ and $\|K\|_T$. h.o.t. stands for higher order term with respect to $\Delta \theta$, that is, there exists positive constant $C$ which depends only on $T$, $A$ and $B$ such that $\|\text{h.o.t.}\| \leq C (\Delta \theta_2)^2$. We can choose that $A, B$ so large and $\Delta \theta_2$ so small such that (3.5) holds with $\overline{K}^m = \overline{K}^m$. In quite same manner, we can choose $A, B$ and $\Delta \theta_2$ such that (3.6) holds with $\underline{K}^m = \underline{K}^m$.

Using induction on $m$ we can show convergence of curvature. From what we proved in section 2, we can choose $A, B$ so that
\[
K^m_j \leq k^m_j \leq K^m_j
\]
for $0 \leq j < n$.

Assume that $K^m_j \leq k^m_j \leq K^m_j$. If we choose $\Delta \theta$ appropriately small, then condition (3.2) in Lemma 3.3 holds with $U = K^m$ and $V = K^m$. In fact, we have
\[
K^m_j = K^m_j + A e^{B_m} \Delta \theta_2 \leq k^m_j + 2 A e^{B T} \Delta \theta_2 \leq k^m_j + \varepsilon k^0_{\min} \leq (1 + \varepsilon) k^m_j,
\]
for appropriately large \( n \geq \frac{C}{\sqrt{\varepsilon}} \), where \( C = C(T, \|V\|_{r}, \min_{\theta \in \mathbb{T}_{n}} \varphi(\theta)) \). At the last inequality we use the fact shown in proof of Theorem 3.4. Therefore

\[
2\alpha \left( K_{j}^{m+1} \right)^{\alpha+1} \cos \Delta \theta \frac{T_{j}^{m+1}}{\Delta \theta_{2}} \leq 2\alpha \lambda_{\alpha}(\varepsilon) \left( K_{j}^{m} \right)^{\alpha+1} \left( \eta_{\max}^{m} \right)^{\alpha+1} < \min \{ \alpha, 1 \} \leq 1.
\]

Hence from (3.4) we can see (3.2) holds, so we have

\[
K_{j}^{m+1} \leq K_{j}^{m+1} \leq K_{j}^{m+1}
\]

for \( 0 \leq j < n \). Then convergence of curvature is proved and this also leads to proof of Theorem 1.2(a).

**Remark 3.5.** It is very essential that

\[
K_{\inf} := \inf_{t \geq 0, \theta \in \mathbb{T}_{n}} \min_{i} K(\theta, t) > 0 \quad \text{and} \quad k_{\inf} := \inf_{m \geq 0, 0 \leq j < n} k_{j}^{m} > 0.
\]

The former inequality is the result of the equality \( K_{\inf} = \min_{t} K(\theta, t) > 0 \) which is proved by the usual comparison principle and the latter inequality is led by the estimate \( k_{\inf} = \min_{t} \) (see proof of Theorem 3.4) and \( \inf_{m \geq 0} k_{\min}^{m} > 0 \). The last inequality \( \inf_{n \geq 1} k_{\min}^{m} > 0 \) holds by virtue of Proposition 2.4 and the way of construction of \( \Gamma_{\Delta}^{0} \) (see Procedure 2.1).

**Proof of Theorem 1.2(b).** We introduce Hausdorff’s distant function. On compact sets \( X_{1} \) and \( X_{2} \), the Hausdorff metric is defined as

\[
D_{H}(X_{1}, X_{2}) := \max \left\{ \sup_{x_{1} \in X_{1}} \inf_{x_{2} \in X_{2}} |x_{1} - x_{2}|, \sup_{x_{2} \in X_{2}} \inf_{x_{1} \in X_{1}} |x_{1} - x_{2}| \right\}.
\]

We define \( \Pi_{m} \) to be the piecewise linear curve constructed by the union of segments on lines tangent to \( \Gamma(t_{m}) \) at points with exterior normal \( -N_{j} \), and so \( \Pi_{0} = \Gamma_{\Delta}^{0} \). We prove this in two steps. First we prove that there exists positive constant \( C_{d}^{1} \) such that \( D_{H}(\Gamma_{\Delta}^{m}, \Pi_{m}) \leq C_{d}^{1}(\Delta \theta)^{2} \). Next by using a usual interpolation argument we can show that \( D_{H}(\Pi_{m}, \Gamma(t_{m})) \leq C_{d}^{2}(\Delta \theta)^{2} \) for some positive constant \( C_{d}^{2} \). This method is a discrete version of the proof of the main theorem in [6].

**Step 1.** Due to Procedure 2.2(c) the maximum distance between the \( j \)th side of \( \Gamma_{\Delta}^{m} \) and \( \Pi_{m} \) can be estimated from above. In fact, since these sides are parallel to each other and \( \Gamma_{\Delta}^{0} = \Pi_{0} \), there exists a positive constant \( C_{d}^{1} \) such that for all \( n \geq n_{v} \),

\[
\left| \sum_{i=0}^{m-1} v_{j}^{i} \tau_{i+1} - \int_{0}^{t_{m}} V(j \Delta \theta, \zeta) d\zeta \right| \leq \sum_{i=0}^{m-1} \tau_{i+1} \left| v_{j}^{i} - V(j \Delta \theta, t_{i}) - \frac{V_{j}(j \Delta \theta, \xi_{j}) \tau_{i+1}}{2} \right|_{\max} \leq C_{d}^{1} t_{m}(\Delta \theta)^{2},
\]

where \( \xi_{j} \) is a suitable value on the mean value theorem and \( |\cdot|_{\max} = \max_{0 \leq j < n} |\cdot|_{j} \).

Therefore \( \Pi_{m} \) lies in a strip of width \( 2C_{d}^{1} t_{m}(\Delta \theta)^{2} \) whose center is \( \Gamma_{\Delta}^{m} \) and there exists \( C_{d}^{1} > 0 \) such that

\[
D_{H}(\Gamma_{\Delta}^{m}, \Pi_{m}) \leq C_{d}^{1} t_{m}(\Delta \theta)^{2} \leq \frac{C_{d}^{1}(\Delta \theta)^{2}}{\sin \left( \frac{\pi}{2 - \Delta \theta} \right)} \leq C_{d}^{1}(\Delta \theta)^{2}.
\]

**Step 2.** By standard interpolation argument we can prove that there exists a positive constant \( C_{d}^{2} \) such that

\[
D_{H}(\Pi_{m}, \Gamma(t_{m})) \leq \operatorname{dist}(Q_{\Pi}, B_{\Pi}) \leq C_{d}^{2}(\Delta \theta)^{2}
\]
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\[
\begin{align*}
\text{Fig. 3.1. The movement on } m\text{th step (outside frame is } \Gamma_\Delta^m, \text{ inside frame is } \Gamma_{\Delta+1}^m, \text{ and } \bullet = \Delta \theta).}
\end{align*}
\]

(see, e.g., [6] for details). Hence the assertion holds by \( C_d := C_1^d + C_2^d \).

**Proof of Corollary 1.3.** As in the proof of Theorem 1.2(b), the piecewise linear curve \( \Pi_m \) lies in a strip of width \( 2 \tilde{C}_1^d t_m (\Delta \theta)^2 \) whose center is \( \Gamma_\Delta^m \).

We define \( \bar{\Pi}_m \) and \( \bar{\Pi}_m \) to be the piecewise linear curve obtained by the union of segments on the outside and inside lines of this strip, respectively. Denote the length for the \( j \)th side of \( \bar{\Pi}_m \) (respectively, \( \bar{\Pi}_m \)) by \( \bar{d}_j^m \) (respectively, \( \bar{d}_j^m \)). Moreover, we set \( L_\Delta^m := \sum_{j=0}^{n-1} \bar{d}_j^m \) and \( \bar{L}_\Delta^m := \sum_{j=0}^{n-1} \bar{d}_j^m \). In the case when \( \eta = 1 \), we define \( A_\Delta^m \) (respectively, \( A_\Delta^m \)) to be the area enclosed by \( \bar{\Pi}_m \) (respectively, \( \bar{\Pi}_m \)).

Taking geometric requirements (Figure 3.1) into account, we can see that \( \bar{d}_j^m \) and \( \bar{d}_j^m \) must satisfy

\[
\begin{align*}
\bar{d}_j^m &= \bar{d}_j^m - 2 \tan \frac{\Delta \theta}{2} \tilde{C}_1^d t_m (\Delta \theta)^2, \\
\bar{d}_j^m &= \bar{d}_j^m - 2 \tan \frac{\Delta \theta}{2} \tilde{C}_1^d t_m (\Delta \theta)^2.
\end{align*}
\]

Therefore for all \( n \geq n_t \) and \( 0 \leq t_m \leq T \)

\[
\max \left\{ L_\Delta^m, \bar{L}_\Delta^m \right\} \leq L_\Delta^m + 2 \tan \frac{\Delta \theta}{2} \tilde{C}_1^d t_m (\Delta \theta)^2 n
\]

\[
\leq L(t_m) + C_1 (\Delta \theta)^2 + 2 \tan \frac{\Delta \theta}{2} \tilde{C}_1^d T (\Delta \theta)^2 n < \infty.
\]

Hence there exist positive constant \( C_a \) and natural number \( n_a := \max\{n_0^a, n_t^a\} \) such that

\[
|A_\Delta^m - A(t_m)| \leq \bar{A}_\Delta^m - A_\Delta^m = 2 \tilde{C}_1^d t_m \sum_j \bar{d}_j^m + \bar{d}_j^m (\Delta \theta)^2
\]

\[
= \tilde{C}_1^d t_m (L_\Delta^m + \bar{L}_\Delta^m) (\Delta \theta)^2 \leq C_a (\Delta \theta)^2
\]

holds for all \( n \geq n_a \) and then the assertion holds.

**4. Numerical examples.** In this section we exhibit several numerical examples. In the first subsection, we examine the reliability of our scheme. Comparison between numerical and exact solutions suggests the convergence rate given in Theorem 1.2 is optimal. In the second subsection, we shall show some interesting examples.
Table 4.1
Computation time: step \( m = 10000 \).

<table>
<thead>
<tr>
<th>n</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>time (second)</td>
<td>0.2</td>
<td>0.4</td>
<td>0.9</td>
<td>1.8</td>
<td>3.5</td>
<td>6.9</td>
</tr>
</tbody>
</table>

All the examples below were computed using double precision on the DEC Alpha Station 500/333 computer at Graduate School of Mathematical Sciences, the University of Tokyo. Table 4.1 indicates the computation time for 10000 steps on this machine.

Before showing our numerical data, let us first explain some basic notions about blow-up. The curve \( \Gamma(t) \) is self-similar if and only if there exists a function \( h(t) \) and a fixed curve \( \Gamma \) such that \( \Gamma(t) = h(t)\Gamma + x_0 \), where \( x_0 \) is a fixed point in \( \mathbb{R}^2 \). In our case, \( h(t) = (T_* - t)^{\frac{\alpha+1}{\alpha}} \) and the corresponding solution of Problem 1.1 is given by \( V(\theta, t) = (T_* - t)^{-\frac{\alpha}{\alpha+1}} \phi(\theta) \). We call the blow-up “of Type I” when the blow-up rate is the same as the self-similar rate, that is,

\[
\lim_{t \to T_*} \max_{\theta \in T_*} V(\theta, t)(T_* - t)^{\frac{\alpha+1}{\alpha}} < \infty.
\]

We call the blow-up “of Type II” when the blow-up rate is faster than the self-similar rate, that is,

\[
\lim_{t \to T_*} \max_{\theta \in T_*} V(\theta, t)(T_* - t)^{\frac{\alpha+1}{\alpha}} = \infty.
\]

We note that the aspect ratio of the curve goes to infinity as \( t \to T_* \) if and only if the blow-up is of Type II [19].

We note that the following types of self-similar embedded curves exist: circle (for any \( \alpha > 0 \)), ellipses (for \( \alpha = \frac{1}{3} \)), oval triangle (for \( 0 < \alpha < \frac{1}{8} \)) and oval \( q \)-polygon (for \( 0 < \alpha < \frac{1}{q^2-1} \)), which can be shown by a standard bifurcation argument.

4.1. Verification. We shall compare numerical solutions with exact solutions.

Shrinking circles for \( \alpha > 0 \). For any \( \alpha > 0 \), every circle is one of the exact solution of Problem (1.1) and shrinks to the center in self-similar way. Then the normal velocity is given by

\[
V = R(t)^{-\alpha} \quad \text{with radius} \quad R(t) = (R(0)^{\alpha+1} - (\alpha + 1)t)^{\frac{1}{\alpha+1}}, \quad \alpha > 0.
\]

From this formula we can compute the blow-up time exactly; it is given by \( T_* = \frac{R(0)^{\alpha+1}}{\alpha+1} \). Hereafter, we consider the case when the initial curve is unit circle \( (R(0) = 1) \).

For several \( \alpha > 0 \) and \( \epsilon = 1 \), we observe the rate of convergence and the differences between exact blow-up time \( T_* \) and approximate blow-up time \( t_* \). From Table 4.2, whose figures indicate the values \( \sup_{0 < j < n} |V(j\Delta\theta, t_m) - v_j^m| \) for \( t_m = \frac{1}{4}T_*, \frac{1}{2}T_*, \) and \( \frac{3}{4}T_*, \) and for \( n = 10, 20, 40, 80, 160 \) and 320, we can see that the rate of convergence is \( O((\Delta\theta)^2) \) in every case. In Figure 4.1, values \( \sup_{0 < j < n} |V(j\Delta\theta, t_m) - v_j^m| \) are plotted versus \( n \) in logarithmic scale. Table 4.3 indicates approximate blow-up times. Here we define approximate blow-up time \( t_* \) by \( t_m \) such that \( v_{\max}^m \approx 10^6 \).

Shrinking ellipses for \( \alpha = \frac{1}{3} \). In the case \( \alpha = \frac{1}{3} \), any ellipses are exact solutions of Problem (1.1). In fact, every ellipse is self-similar curve if and only if \( \alpha = \frac{1}{3} \) and
we can check that the exact solution is given by
\[ V(\theta, t) = (T_\star - t)^{-\frac{1}{2}} \varphi(\theta) \]
and the blow-up time is given by \( T_\star = \frac{4}{3} (ab)^{\frac{2}{3}} \), where \( a \) and \( b \) are the initial short and long radiuses, respectively. We observe that the evolution which starts from the initial curve given by
\[ X_0(\theta) = X_E(\theta; a, b) := R(\theta)^{-\frac{1}{2}} (a^2 \cos \theta, b^2 \sin \theta), \quad \theta \in \mathbb{T}_1 \] with \( (a, b) = (1, 3) \), where \( R(\theta) = a^2 \cos^2 \theta + b^2 \sin^2 \theta \). Then \( T_\star = \frac{3}{4} 3^{\frac{2}{3}} \approx 1.560 \).
Fig. 4.1. Circle: \( \alpha = \frac{1}{10}, \frac{1}{4}, \frac{1}{3}, 1, 2, 4, \varepsilon = 1, \) sup \( j |V(j\Delta\theta, t_m) - \nu_j^m| \) vs. \( n \) in log\(_{10}\)-scale. Here \( t_m = \frac{1}{4}T_\ast \).

Table 4.3

<table>
<thead>
<tr>
<th>( n )</th>
<th>( T_\ast )</th>
<th>( \frac{1}{10} )</th>
<th>( \frac{1}{4} )</th>
<th>( \frac{1}{3} )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.91327</td>
<td>0.80828</td>
<td>0.75976</td>
<td>0.67825</td>
<td>0.51223</td>
<td>0.33734</td>
<td>0.20660</td>
</tr>
<tr>
<td>(( n=10 ))</td>
<td>(662)</td>
<td>(527)</td>
<td>(397)</td>
<td>(284)</td>
<td>(577)</td>
<td>(2315)</td>
</tr>
<tr>
<td>0.91014</td>
<td>0.80207</td>
<td>0.75825</td>
<td>0.66957</td>
<td>0.53078</td>
<td>0.33346</td>
<td>0.20601</td>
</tr>
<tr>
<td>(5983)</td>
<td>(2115)</td>
<td>(1584)</td>
<td>(1124)</td>
<td>(725)</td>
<td>(2256)</td>
<td>(9033)</td>
</tr>
<tr>
<td>0.90435</td>
<td>0.80156</td>
<td>0.75061</td>
<td>0.66739</td>
<td>0.50077</td>
<td>0.33359</td>
<td>0.20302</td>
</tr>
<tr>
<td>(23986)</td>
<td>(8464)</td>
<td>(6336)</td>
<td>(4483)</td>
<td>(281)</td>
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<td>(35909)</td>
</tr>
<tr>
<td>0.90138</td>
<td>0.80013</td>
<td>0.75015</td>
<td>0.66685</td>
<td>0.50019</td>
<td>0.33340</td>
<td>0.20301</td>
</tr>
<tr>
<td>(95999)</td>
<td>(33860)</td>
<td>(25340)</td>
<td>(17921)</td>
<td>(13851)</td>
<td>(2315)</td>
<td>(9033)</td>
</tr>
<tr>
<td>0.90091</td>
<td>0.80003</td>
<td>0.75004</td>
<td>0.66671</td>
<td>0.50005</td>
<td>0.33335</td>
<td>0.20000</td>
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<tr>
<td>(384052)</td>
<td>(135446)</td>
<td>(101357)</td>
<td>(71674)</td>
<td>(143356)</td>
<td>(2293512)</td>
<td></td>
</tr>
<tr>
<td>0.90069</td>
<td>0.80001</td>
<td>0.75001</td>
<td>0.66668</td>
<td>0.50001</td>
<td>0.33334</td>
<td>0.20000</td>
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<tr>
<td>(1536261)</td>
<td>(541789)</td>
<td>(405427)</td>
<td>(286684)</td>
<td>(573376)</td>
<td>(2293512)</td>
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</tr>
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</table>

Table 4.4 indicates the values sup \( |V(t_m,j\Delta\theta) - \nu_j^m| \) for \( t_m = \frac{1}{10}T_\ast, \frac{1}{4}T_\ast \), \( \frac{1}{3}T_\ast \), and \( \frac{2}{3}T_\ast \), and for \( n = 10, 20, 40, 80, 160, \) and \( 320 \) and for \( \varepsilon = 0.25, 0.5, 1 \) and 2. In Figure 4.2, values sup \( |V(j\Delta\theta, t_m) - \nu_j^m| \) are plotted versus \( n \) in logarithmic scale for \( \varepsilon = 1 \). It can be observed that the rate of convergence is \( O((\Delta\theta)^2) \) in every case.

Shrinking curves for \( \alpha = 1 \) and \( \eta = 1 \). In this case, if the initial area is \( A(0) \), then the blow-up time is given by \( T_\ast = \frac{A(0)}{2\pi} \). For the initial curve which is given by \( X_0(\theta) = X_E(\theta; 1, 3) \), we examine the differences between exact blow-up time \( T_\ast = 1.5 \) and approximate blow-up time \( t_\ast \). Table 4.5 indicates \( t_\ast \) for several partition numbers \( n \). Here approximate blow-up time is determined in the same manner as the previous case.

4.2. Simulation. In this subsection, we give some numerical examples for the case shrinking ellipses, oval polygons, and extinction of loop.

From the previous subsection, we can observe the rate of convergence is \( O((\Delta\theta)^2) \).
for $n \geq 80$ when $\varepsilon = 1$. So in this subsection all examples below are calculated with $n \geq 80$ and $\varepsilon = 1$.

**Shrinking ellipses.** In Figure 4.3, we show the evolutions starting from the same ellipse for several $\alpha > 0$. The initial curve is given by $X_0(\theta) = X_\theta(\theta; 1, 3)$. We set the parameter $n = 80$. Table 4.6 indicates the time $t_m$ at which curves in Figure 4.3
Table 4.5

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>2</th>
<th>1</th>
<th>0.5</th>
<th>0.25</th>
</tr>
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<tr>
<td>$T^*_{\Delta}$ exact</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
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<tr>
<td>$n = 10$</td>
<td>1.65696</td>
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<td>1.67216</td>
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<tr>
<td>($m = 814$)</td>
<td>(358)</td>
<td>(199)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$20$</td>
<td>1.53230</td>
<td>1.53339</td>
<td>1.53529</td>
<td>1.53711</td>
</tr>
<tr>
<td>($m = 3254$)</td>
<td>(1443)</td>
<td>(809)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$40$</td>
<td>1.50841</td>
<td>1.50872</td>
<td>1.50916</td>
<td>1.50960</td>
</tr>
<tr>
<td>($m = 12993$)</td>
<td>(5772)</td>
<td>(3244)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$80$</td>
<td>1.50211</td>
<td>1.50219</td>
<td>1.50230</td>
<td>1.50241</td>
</tr>
<tr>
<td>($m = 51959$)</td>
<td>(23090)</td>
<td>(12985)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$160$</td>
<td>1.50053</td>
<td>1.50055</td>
<td>1.50058</td>
<td>1.50060</td>
</tr>
<tr>
<td>($m = 207821$)</td>
<td>(92362)</td>
<td>(51951)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$320$</td>
<td>1.50013</td>
<td>1.50014</td>
<td>1.50014</td>
<td>1.50015</td>
</tr>
<tr>
<td>($m = 831272$)</td>
<td>(369451)</td>
<td>(207814)</td>
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<td></td>
</tr>
</tbody>
</table>

Table 4.6

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0 = 0.00000$</td>
<td>$t_0 = 0.00000$</td>
<td>$t_0 = 0.00000$</td>
<td>$t_0 = 0.00000$</td>
</tr>
<tr>
<td>$t_{1064} = 0.280855$</td>
<td>$t_{1317} = 0.33257$</td>
<td>$t_{1387} = 0.23622$</td>
<td>$t_{4723} = 0.16167$</td>
</tr>
<tr>
<td>$t_{2569} = 0.55129$</td>
<td>$t_{2921} = 0.64414$</td>
<td>$t_{2502} = 0.43867$</td>
<td>$t_{5044} = 0.42997$</td>
</tr>
<tr>
<td>$t_{5044} = 0.81118$</td>
<td>$t_{4982} = 0.93140$</td>
<td>$t_{3560} = 0.83084$</td>
<td>$t_{11487} = 0.74059$</td>
</tr>
<tr>
<td>$t_{9275} = 1.05861$</td>
<td>$t_{7872} = 1.18890$</td>
<td>$t_{6483} = 1.12902$</td>
<td>$t_{14642} = 1.08467$</td>
</tr>
<tr>
<td>$t_{19345} = 1.28951$</td>
<td>$t_{12746} = 1.40707$</td>
<td>$t_{5905} = 1.38525$</td>
<td>$t_{8257} = 1.32351$</td>
</tr>
<tr>
<td>$t_{216644} = 1.49003$</td>
<td>$t_{34063} = 1.55737$</td>
<td>$t_{10153} = 1.50213$</td>
<td>$t_{38659} = 1.36692$</td>
</tr>
</tbody>
</table>

Fig. 4.3. Initial curve: ellipses. (a) $\alpha = \frac{1}{4}$, (b) $\alpha = \frac{1}{3}$, (c) $\alpha = 1$, and (d) $\alpha = 2$.

and 4.4 are plotted.

In Figure 4.4 we show self-similar scaled curves which are obtained by magnifying curves in Figure 4.3 by $(t_* - t_m)^{-\alpha/\varepsilon}$. We note that by a similar argument as in the proof of Theorem 3.4 we have $T_\Delta - t_m \leq \frac{L^\alpha_{\Delta}}{2\pi(k_{\min}^\alpha)^\alpha}$. So we choose $t_m$ when $\frac{L^\alpha_{\Delta}}{2\pi(k_{\min}^\alpha)^\alpha} \approx 10^{-3}$ as the approximate blow-up time $t_*$. In Figure 4.3(a), which is the case $\alpha = \frac{1}{4} < \frac{1}{3}$, we can observe that aspect ratio...
of the curve becomes large. As mentioned in section 1, if the case $\alpha < \frac{1}{3}$, then there exists a Type II blow-up solution to Problem 1.1 and an aspect ratio of the curve which corresponds to blow-up solution of Type II goes to infinity as $t$ tends to $T_\ast$.

In Figure 4.3(b), which is the case $\alpha = \frac{1}{3}$, we observe that the ellipse retains its shape just before shrinking to a point. In Figure 4.3(c), which is the case $\alpha = 1$, the ellipse shrinks to a “round point”: its asymptotic shape just before it disappears is a circle. This example coincides with the mathematical result [5]. In Figure 4.3(d),
which is the case $\alpha = 2 > 1$, the ellipse rapidly shrinks to a round point.

**Shrinking oval polygons.** In Figure 4.5, we show the evolutions starting from “oval (regular)polygons” which have $q$ linear-like slopes and $q$ round corners. We give data of oval (regular)polygons. This data is obtained as a numerical solution of anisotropic curvature flow: $V(\theta, t) = (1 + 0.9 \cos(q\theta))K(\theta, t)$ which starts from initial circle with radius 3. We choose the approximated curve for the time $t \approx 3.857$ as the initial data. We set parameter $n = 80$. Table 4.7 indicates the time $t_m$ at which curves in Figure 4.5 and 4.6 are plotted.

In Figure 4.6, we show self-similar scaled curves which are obtained by magnifying curves in Figure 4.5 by $(t_* - t_m)^{-\frac{1}{\alpha+1}}$. We choose $t_*$ in the same manner as Figure 4.4.

In Figure 4.5(a), which is the case $\alpha = 1$ and initial curve is oval triangle ($q = 3$), rapid shrinking (to a circular point) is observed.
Table 4.8

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$t_0 = 0.00000$</td>
<td>$t_0 = 0.00000$</td>
</tr>
<tr>
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<td>$t_{354} = 0.09480$</td>
<td>$t_{357} = 0.09222$</td>
</tr>
<tr>
<td>$t_{973} = 0.11479$</td>
<td>$t_{1062} = 0.09818$</td>
<td>$t_{100} = 0.09537$</td>
</tr>
<tr>
<td>$t_{2076} = 0.11490$</td>
<td>$t_{2223} = 0.09829$</td>
<td>$t_{212} = 0.09547$</td>
</tr>
</tbody>
</table>

Figures 4.5(b), (c), and (d) indicate the case when the initial curve is the oval triangle ($q = 3, \alpha \leq \frac{1}{11}$), the oval rectangle ($q = 4, \alpha = \frac{1}{27} \leq \frac{1}{11}$), and the oval pentagon ($q = 5, \alpha = \frac{1}{30} \leq \frac{1}{27}$), respectively. We can observe that these oval polygons shrink in an asymptotically self-similar way.

**Extinction of loop.** In Figure 4.7(a), (b), and (c) we treat the case when the initial curves have self-intersections. We set parameter $\alpha = 1$. In Figure 4.7(a), we set partition number $n = 160$, and in Figure 4.7(b) and (c) we set $n = 240$. Table 4.8 indicates the time $t_m$ at which curves in Figure 4.7 are plotted.

Figure 4.7(a) shows that single loop shrinks and may have cusp. This situation is studied in [3]. In Figure 4.7(b), initial curve has slightly larger left loop than right loop. We observe that smaller loop shrinks rapidly. In Figure 4.7(c), the initial loop is in loop, the smallest loop is rapidly extinct.

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**REFERENCES**


